Mass concentration in rescaled first order integral functionals

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We consider first order local minimization problems $\min \int_{\mathbb{R}^N} f(x_0, u, \nabla u)$ over non-negative Sobolev functions u satisfying a mass constraint $\int_{\mathbb{R}^N} u = m$. We prove that the minimal energy function $H(x_0, m)$ is always concave in m, and that relevant rescalings of the energy, depending on a small parameter ε, Γ-converge in the weak topology of measures towards the H-mass, defined for atomic measures $\sum_i m_i \delta_{x_i}$ as $\sum_i H(x_i, m_i)$. The Γ-convergence result holds under mild assumptions on the Lagrangian, and covers several situations including homogeneous H-masses in any dimension $N \geq 2$ for exponents above a critical threshold, and all concave H-masses in dimension $N = 1$. Our result yields in particular the concentration of Cahn-Hilliard fluids into droplets, and is related to the approximation of branched transport by elliptic energies.

Contents

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Notation

$B_r(x)$	open ball of radius r centered at x;
B_r	open ball $B_r(0)$;
$\mathcal{M}(\mathbb{R}^N)$	set of finite signed Borel measures on $\mathbb{R}^N$;
$\mathcal{M}_+(\mathbb{R}^N)$	set of finite positive Borel measures on $\mathbb{R}^N$;
$\tau_x\mu$	Borel measure $A \mapsto \mu(A - x)$ if $\mu \in \mathcal{M}(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$;
$c_B\mu$	Borel measure $\tau_{-x}(\mu \llcorner B)$ if B is the ball $B_r(x)$;
$\mu_\ell \xrightharpoonup{\mathcal{C}_0'} \mu$	weak convergence of measures, i.e. weak-$\star$ convergence in duality with the space $\mathcal{C}_0(\mathbb{R}^N)$ of continuous functions vanishing at infinity;
$\mu_\ell \xrightharpoonup{\mathcal{C}_b'} \mu$	narrow convergence of measures, i.e. weak-$\star$ convergence in duality with thhe space of continuous and bounded function $\mathcal{C}_b(\mathbb{R}^N)$;
Σ	set of increasing maps $\sigma : \mathbb{N} \to \mathbb{N}$;
$\sigma_1 \preceq \sigma_2$	$\sigma_1, \sigma_2 \in \Sigma$ are such that $\sigma_1(\llbracket n, +\infty\rrbracket) \subseteq \sigma_2(\mathbb{N})$ for some $n \in \mathbb{N}$.

1 Introduction

1.1 Setting

Let $N \in \mathbb{N}^*$ and let $f : \mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}_+$ be a Borel function. Consider the following energy functional, defined for any fixed $x \in \mathbb{R}^N$ on the set of finite positive Borel measures $\mathcal{M}_+(\mathbb{R}^N)$ on $\mathbb{R}^N$ by

$$\mathcal{E}_f^x(u) = \begin{cases} \displaystyle\int_{\mathbb{R}^N} f(x, u(y), \nabla u(y))\,\mathrm{d}y & \text{if } u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}_+), \\ +\infty & \text{otherwise.} \end{cases} \tag{1.1}$$

The minimization of this energy energy under a mass constraint gives rise to the notion of minimal cost function, defined by

$$H_f(x,m) := \inf\left\{\mathcal{E}_f^x(u) \ :\ u\in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{R}_+) \text{ such that } \int_{\mathbb{R}^N} u = m\right\} \in [0,+\infty]. \quad (1.2)$$

As a preliminary result, which deserves interest on its own, we will establish the following:

Theorem 1.1. *Let $x\in\mathbb{R}^N$. The map $m\mapsto H_f(x,m)$ is concave non-decreasing on $(0,+\infty)$, and if we further assume that $f(x,0,0)=0$ and $H_f(x,\cdot)\not\equiv +\infty$ on $(0+\infty)$, then it is also continuous on $[0,+\infty)$ with $H_f(x,0)=0$.*

The proof is very simple and works with no further assumptions on f, and even in a slightly more general situation as stated in Theorem 2.1.

Our main purpose is to prove that under some conditions, if $(f_\varepsilon)_{\varepsilon>0}$ is a family of functions $f_\varepsilon : \mathbb{R}^N\times\mathbb{R}_+\times\mathbb{R}^N\to\mathbb{R}_+$ converging pointwise to f as $\varepsilon\to 0$, then the rescaled energy functionals $\mathcal{E}_\varepsilon$, defined for each $\varepsilon>0$ on $\mathcal{M}_+(\mathbb{R}^N)$ by

$$\mathcal{E}_\varepsilon(u) = \begin{cases}\displaystyle\int_{\mathbb{R}^N} f_\varepsilon(x,\varepsilon^N u(x),\varepsilon^{N+1}\nabla u(x))\varepsilon^{-N}\,\mathrm{d}x & \text{if } u\in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{R}_+),\\ +\infty & \text{otherwise,}\end{cases} \quad (1.3)$$

Γ-converge as $\varepsilon\to 0$, for the narrow or weak convergence of measures, to the H_f-mass, defined on $\mathcal{M}_+(\mathbb{R}^N)$ by (see Definition 2.5):

$$\mathbf{M}^{H_f}(u) := \sum_{i\in I} H_f(x_i,m_i) + \int_{\mathbb{R}^N} H_f'(x,0)\,\mathrm{d}u^d(x).$$

where $u = u^a + u^d$ is the decomposition of u into its atomic part $u^a = \sum_{i\in I} m_i\delta_{x_i}$ where $m_i = u(\{x_i\})$ for every $i\in I\subseteq\mathbb{N}$, and its diffuse part u^d, and $H_f'(x,0) = \lim_{m\to 0^+}\frac{H_f(x,m)}{m}\in[0,+\infty]$.

This kind of singular limit in integral functionals is reminiscent of several variational models with physical relevance which have been the object of intensive mathematical analysis, such as Cahn-Hilliard fluids with concentration on droplets [BDS96] or on singular interfaces [MM77], toy models for micromagnetism and liquid crystals like Aviles-Giga [AG99] and Landau-de Gennes [BPP12], or Ginzburg-Landau theory of supraconductivity [Hél94].

The fact that $\mathbf{M}^{H_f}$ is expected to be the Γ-limit of $\mathcal{E}_\varepsilon$ is due to the following observation: if $B_r(x_0)\subseteq\mathbb{R}^N$ and $u_\varepsilon(x) := \varepsilon^{-N}v_\varepsilon(\varepsilon^{-1}(x-x_0))$, then $\int_{B_r(x_0)} u_\varepsilon = \int_{B_{r/\varepsilon}} v_\varepsilon$ and

$$\int_{B_r(x_0)} f_\varepsilon(x,\varepsilon^N u_\varepsilon(x),\varepsilon^{N+1}\nabla u_\varepsilon(x))\varepsilon^{-N}\,\mathrm{d}x = \int_{B_{r/\varepsilon}} f_\varepsilon(x_0+\varepsilon y, v_\varepsilon(y),\nabla v_\varepsilon(y))\,\mathrm{d}y,$$

so that the energy contribution of a mass $m\geq 0$ contained in a ball $B_r(x_0)$ should be of the order of $H_f(x_0,m)$, where r is arbitrary.

Nevertheless, it is not true in general that $\mathbf{M}^{H_f}$ is the Γ-limit of the functionals $\mathcal{E}_\varepsilon$ (see Section 1.3 below). We will need a couple of assumptions on f and f_ε detailed in the next section.

1.2 Assumptions and main result

Our first two assumptions are rather standard and guarantee the sequential lower semicontinuity of the functionals $\mathcal{E}_f^x$,

(H1) f is lower semicontinuous on $\mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}^N$,

(H2) $f(x, u, \cdot)$ is convex for every $x \in \mathbb{R}^N, u \in \mathbb{R}_+$.

We also need continuity in the spatial variable:

(H3) $f(\cdot, u, \xi)$ is continuous for every $u \in \mathbb{R}_+, \xi \in \mathbb{R}^N$.

Next, we need a compactness assumption which ensures relative compactness in the weak topology of $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$ for sequences of bounded energy $\mathcal{E}_f^x$ and bounded mass; it will also be needed in obtaining lower bounds for the energy (see Proposition 3.8):

(H4) there exist $\alpha, \beta \in (0, +\infty)$, $p \in (1, +\infty)$ such that for all $(x, u, \xi) \in \mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}^N$,

$$f(x, u, \xi) \geq \alpha |\xi|^p - \beta u.$$

We also impose a condition on the slope of $f(x, \cdot, \xi)$ at the origin which will be needed in order to identify the initial slope of $H_f(x, \cdot)$ (see Section 2.3), and rules out some non-trivial scale invariant Lagrangians for which the expected Γ-convergence result fails (see Section 1.3),

(H5) for every $x_0 \in \mathbb{R}^N$,

$$f'_-(x_0, 0, 0) \coloneqq \liminf_{(x,u,\xi)\to(x_0,0^+,0)} \frac{f(x, u, \xi)}{u} \geq \limsup_{u\to 0^+} \sup_{|\xi|\leq 1} \frac{f(x_0, u, \rho(u)\xi)}{u}, \tag{1.4}$$

with $\rho \equiv 0$ if $N = 1$ and for some $\rho \in \mathcal{C}((0, 1], (0, +\infty))$ satisfying

$$\int_0^1 \Big(\int_y^1 \frac{\mathrm{d}t}{\rho(t)} \Big)^N \mathrm{d}y < +\infty \quad \text{if } N \geq 2.$$

Last of all, we need the family of functions $f_\varepsilon : \mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}_+$ to converge towards f in a suitable sense, namely, we assume

(H6) $f_\varepsilon \uparrow f$ and $f'_{\varepsilon,-}(\cdot, 0, 0) \uparrow f'_-(\cdot, 0, 0)$ as $\varepsilon \to 0$.

Notice that this assumption is empty if f_ε does not depend on ε.

Our main result is the following:

Theorem 1.2. *If $(f_\varepsilon)_{\varepsilon>0}$ satisfies (H6) with each f_ε satisfying (H1)–(H4) and the limit f satisfying (H5), then $\mathbf{M}^{H_f}$ is the Γ-limit as $\varepsilon \to 0$ of the functionals $\mathcal{E}_\varepsilon$, defined in (1.3), for both the weak convergence and the narrow convergence of measures.*

In particular, as a Γ-limit, the functional $\mathbf{M}^{H_f}$ must be lower semicontinuous for the weak convergence of measures (and so for the narrow convergence as well). This implies that H_f is lower semicontinuous on $\mathbb{R}^N \times \mathbb{R}_+$ (see Proposition 2.7).

We point out that for the $\Gamma - \limsup$, we need weaker assumptions on f_ε and f (see Proposition 4.2), which will be useful for some applications (see Section 5.5).

1.3 Examples, counterexamples and applications

We start with a counterexample, justifying the importance of (H5), and we then provide several examples satisfying our assumptions.

Scale invariant Lagrangians. In the particular case where $f_\varepsilon \equiv f$ and $f(x,u,\xi) = u^{-p(1-\frac{1}{p^\star})}|\xi|^p$, with $p \in (1,N)$ and $p^\star = \frac{pN}{N-p}$, we find that

$$\mathcal{E}_\varepsilon(u) = \int_{\mathbb{R}^N} f(x, \varepsilon^N u, \varepsilon^{N+1}\nabla u)\varepsilon^{-N} = \int_{\mathbb{R}^N} u^{-p(1-\frac{1}{p^\star})}|\nabla u|^p = \mathcal{E}_f(u),$$

i.e. the rescaled energies $\mathcal{E}_\varepsilon$ do not depend on $\varepsilon > 0$. A scaling analysis also shows that the associated cost function satisfies $H_f(m) = m^{1-\frac{p}{N}} H_f(1)$. Moreover, it can be seen that $0 < H_f(1) < +\infty$, which implies that the Γ-limit of $\mathcal{E}_\varepsilon$, which is nothing but the lower semicontinuous relaxation of $\mathcal{E}_f$, does not coincide with $\mathbf{M}_{H_f}$. Considering the perturbation of f given by $\tilde{f}(x,u,\xi) = f(x,u,\xi) + |\xi|^p$, we find a Lagrangian satisfying all our assumptions except (H5) (note that $|\xi|^p$ is needed in (H4)), and such that the associated rescaled energies do not Γ-converge to $\mathbf{M}_{H_{\tilde{f}}}$ (see Section 5.1). Hence, an assumption like (H5) is required in our Γ-convergence result. We will even see that the lower semicontinuity of H_f and $\mathbf{M}_{H_f}$ is not guaranteed without (H5).

Concave H-masses in dimension one. Consider the energy

$$\mathcal{E}_f(u) = \int_{\mathbb{R}^N} |\nabla u|^2 + c(u) \quad \text{with Lagrangian} \quad f(x,u,\xi) = |\xi|^2 + c(u).$$

In dimension $N = 1$, it is shown in [Wir19] that for any concave continuous function H with $H(0) = 0$, there exists a suitable $c \geq 0$ such that $H_f = H$. As explained in Section 5.2, Theorem 1.2 implies that the rescaled energies

$$\mathcal{E}_\varepsilon(u) = \int_{\mathbb{R}^N} f(\varepsilon^N u, \varepsilon^{N+1}\nabla u)\varepsilon^{-N} \tag{1.5}$$

Γ-converge to $\mathbf{M}^H$, leading to an elliptic approximation of any concave H-mass in dimension one. However, in dimension $N \geq 2$, we have no positive or negative answer to the inverse problem, consisting in finding f such that $H = H_f$ for a given H.

Homogeneous H-masses in any dimension. We consider variants of (5.1) with an additional sublinear term, so as to satisfy our assumptions:

$$\mathcal{E}_f(u) = \int_{\mathbb{R}^N} f(u, \nabla u) = \int_{\mathbb{R}^N} |\nabla u|^p + u^s. \tag{1.6}$$

The rescaled energies as set in (1.5) Γ-converge to a non-trivial multiple of some α-mass $\mathbf{M}^\alpha := \mathbf{M}^{t\mapsto t^\alpha}$ for every $s \in (-p', 1]$, and $\alpha = \frac{1-\frac{s}{p}+\frac{s}{N}}{1-\frac{s}{p}+\frac{1}{N}}$ ranges over $\left(1 - \frac{3}{N+2}, 1\right]$ when s, p vary in their respective range and $N \geq 2$. More cases, with details, are given in Section 5.3.

Cahn-Hilliard approximations of droplets models. Following the works of [BDS96; Dub98], we consider the functionals

$$\mathcal{W}_\varepsilon(u) = \int_{\mathbb{R}^N} \varepsilon^{-\rho}(W(u) + \varepsilon|\nabla u|^2), \tag{1.7}$$

where $W(t) \sim_{t\to+\infty} t^s$ for some exponent $s \in (-2, 1)$. As shown in Section 5.5, we way rewrite these functionals to fit our general framework, and recover known Γ-convergence results, under slightly more general assumptions, as stated in Theorem 5.1. The Γ-limit is a nontrivial multiple of the α-mass with $\alpha = \frac{1-s/2+s/N}{1-s/2+1/N}$.

Elliptic approximations of Branched Transport. The energy of Branched Transport (see [BCM09] for an account of the theory), in its Eulerian formulation, is an H-mass defined this time on vector measures w whose divergence is also a measure,

$$\mathbf{M}_1^H(w) := \int_\Sigma H(x, \theta(x))\,\mathrm{d}\mathcal{H}^1(x) + \int_{\mathbb{R}^d} H'(x, 0)\,\mathrm{d}|w^\perp|, \tag{1.8}$$

where $w = \theta\xi \cdot \mathcal{H}^1 \llcorner \Sigma + w^\perp$ is the decomposition of w into its 1-rectifiable and 1-diffuse parts (see Section 5.4 for more details). An elliptic approximation of Modica-Mortola type has been introduced in [OS11] for $H(m) = m^\alpha, \alpha \in (0, 1)$, and their Γ-convergence result in dimension $d = 2$ has been extended to any dimension in [Mon15] by a slicing method which relates the energy of w to the energy of its slicings. The same slicing method, together with Theorem 1.2, would allow to prove the Γ-convergence of the functionals

$$\mathcal{E}_\varepsilon(w) = \begin{cases} \int_{\mathbb{R}^d} f_\varepsilon(x, \varepsilon^{d-1}|w|(x), \varepsilon^d|\nabla w|(x))\varepsilon^{1-d}\,\mathrm{d}x & \text{if } w \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^d), \\ +\infty & \text{otherwise,} \end{cases} \tag{1.9}$$

toward $\mathbf{M}_1^{H_f}$ for Lagrangians $f_\varepsilon \to f$ satisfying (H1)–(H6), thus covering a wide range of concave H-masses.

1.4 Structure of the paper

In Section 2, we prove the concavity of the cost function H_f with respect to the mass variable m in full generality (Theorem 2.1), we establish useful properties of general H-masses, and we identify the slope at the origin of H_f in terms of f under our assumption (Proposition 2.8 and Proposition 2.9). In Section 3, we apply a concentration-compactness principle to provide a profile decomposition theorem for sequences of positive measures (Theorem 3.2), which is used to obtain our main lower bound for the energy $\mathcal{E}_f$ (Proposition 3.9) and also yields an existence criterion for profiles with minimal energy under a mass constraint (Proposition 3.11). Section 4 is dedicated to proving lower and upper bounds on the rescaled energies $\mathcal{E}_\varepsilon$ (Proposition 4.1 and Proposition 4.2) that imply in particular our main Γ-convergence result (Theorem 1.2). Last of all, in Section 5, we provide several examples of energy functionals that fall into our framework, as summarized in the previous section.

2 Minimal cost function and H-mass

In this section, we study the properties of general H-masses, of costs H_f associated with general Lagrangians f, and we relate the slope of H_f at $m = 0$ to that of f at $(u, \xi) = (0, 0)$ in the variable u, under particular conditions.

2.1 Concavity and lower semicontinuity of the cost function

Our concavity result stated in Theorem 1.1 is a particular case of:

Theorem 2.1. *Let $f : \mathbb{R} \times \mathbb{R}^N \to [0, +\infty]$ be Borel measurable and for every $m \in \mathbb{R}$,*

$$H(m) := \inf \left\{ \mathcal{E}(u) := \int_{\mathbb{R}^N} f(u, \nabla u) \ : \ u \in L^1 \cap W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N), \int_{\mathbb{R}^N} u = m \right\}. \tag{2.1}$$

Then, H is concave non-decreasing on $(0, +\infty)$. In particular, H is either identically $+\infty$ or continuous on $(0, +\infty)$. In the latter case, if we further assume that $f(0, 0) = 0$, then H is continuous on $[0, +\infty)$ with $H(0) = 0$.

Naturally, a similar statement holds on $(-\infty, 0)$ (consider the change of functions $u \to -u$). Considering Lagrangians f taking infinite values, the previous situation covers the case where we have a constraint $(u, \nabla u) \in A$, where $A \subseteq \mathbb{R} \times \mathbb{R}^N$ is Borel measurable. In particular, we can consider the pointwise constraint $u \geq 0$ a.e., as in Theorem 1.1.

Proof. We first prove that H is concave on $(0, +\infty)$. Let $m > 0$ and $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} u = m$. We pick a non-zero vector $v \in \mathbb{R}^N$ and for every $t \in \mathbb{R}$, we set $u^t(\cdot) = u(\cdot + tv)$ and

$$u \wedge u^t(\cdot) = \inf\{u(\cdot), u^t(\cdot)\}, \quad u \vee u^t(\cdot) = \sup\{u(\cdot), u^t(\cdot)\}.$$

We have $u \wedge u^t + u \vee u^t = u + u^t$. Hence

$$\int_{\mathbb{R}^N} u \wedge u^t + \int_{\mathbb{R}^N} u \vee u^t = 2 \int_{\mathbb{R}^N} u = 2m. \tag{2.2}$$

Moreover, it is standard that $u \wedge u^t = u - (u^t - u)_- \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$ with $\nabla(u \wedge u^t) = \nabla u$ a.e. in $\{u \leq u^t\}$ and $\nabla(u \wedge u^t) = \nabla u^t$ a.e. in $\{u > u^t\}$. Since $u \vee u^t = u + u^t - u \wedge u^t$, we have similar identities for $u \vee u^t$, and we obtain

$$\mathcal{E}(u \wedge u^t) + \mathcal{E}(u \vee u^t) = \mathcal{E}(u) + \mathcal{E}(u^t) = 2\mathcal{E}(u). \tag{2.3}$$

Now, let $M : t \mapsto \int_{\mathbb{R}^N} u \wedge u^t$. In view of (2.2), (2.3), and by definition of H, we have proved

$$H(M(t)) + H(2m - M(t)) \leq 2\mathcal{E}(u). \tag{2.4}$$

Now, by continuity of translations in L^1 and since the map $(x, y) \mapsto x \wedge y$ is Lipschitz on $\mathbb{R}^2$, we have that M is continuous on $\mathbb{R}$ with $M(0) = m$. Moreover $\lim_{t\to\infty} M(t) \leq 0$.

Indeed, for every $R > 0$, $\int_{B_R} |u^t| = \int_{B_R(tv)} |u| \to 0$ as $|t| \to +\infty$ by integrability of u. Hence, $u^t \to 0$ locally in measure in $\mathbb{R}^N$ as $|t| \to +\infty$ and, by dominated convergence,

$$M(t) = \int_{\{u<u^t\}} u + \int_{\{u \geq u^t\}} u^t = \int_{\{u<u^t\}} u + \int_{\{u^{-t} \geq u\}} u \xrightarrow[t\to\infty]{} 2\int_{\{u<0\}} u \leq 0.$$

So, by the intermediate value theorem $M(\mathbb{R}) \supseteq (0, m]$. Hence, we have proved $H(\theta) + H(2m-\theta) \leq 2\mathcal{E}(u)$ for every $\theta \in (0, m]$. Taking the infimum over u such that $\int_{\mathbb{R}^N} u = m$, we obtain

$$\frac{H(\theta) + H(2m - \theta)}{2} \leq H(m), \quad \forall \theta \in (0, m],$$

that is, H is midpoint concave on $(0, +\infty)$. Since H is also bounded below (by 0), we can deduce that H is concave $(0, +\infty)$ (see [RV73, Section 72]).

We now justify that if $H(m) < +\infty$ for some $m > 0$ and if $f(0, 0) = 0$, then $\lim_{m\to 0^+} H(m) = H(0) = 0$. By concavity, this will imply that H is finite, continuous and non-decreasing on $[0, +\infty)$. Taking $u = 0$ in the definition of H immediately yields $H(0) = 0$. Now, let $u \in L^1 \cap W^{1,1}_{\text{loc}}(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} u = m > 0$ and $\mathcal{E}(u) < +\infty$. Up to replacing u by $u \vee 0$ and chaging m, one can assume that $u \geq 0$ almost everywhere. Let

$$t_* := \sup\{t \geq 0 \ : \ M(t) > 0\} \in [0, +\infty], \quad \text{where} \quad M(t) = \int_{\mathbb{R}^N} u \wedge u^t.$$

Since M is continuous with $M(0) = \int_{\mathbb{R}^N} u > 0$ and $\lim_{t\to+\infty} M(t) = 0$ as seen above, we have that $t_* \in (0, +\infty]$ and $\lim_{t\to t_*} M(t) = 0$. Moreover, if $t_* = +\infty$, since $u^t \to 0$ locally in measure, by dominated convergence,

$$\limsup_{m\to 0^+} H(m) \leq \limsup_{t\to(t_*)^-} \mathcal{E}(u \wedge u^t) = \limsup_{t\to(t_*)^-} \int_{\{u<u^t\}} f(u, \nabla u) + \int_{\{u^{-t}\geq u\}} f(u, \nabla u) = 0.$$

If $t_* < +\infty$, we have $u \wedge u^{t_*} = 0$ a.e. and $u^t \to u^{t_*}$ locally in measure as $t \to t_*$ by continuity of translation in L^1. Hence,

$$\begin{aligned}
\limsup_{m\to 0^+} H(m) \leq \limsup_{t\to(t_*)^-} \mathcal{E}(u \wedge u^t) &= \limsup_{t\to(t_*)^-} \int_{\{u<u^t\}} f(u, \nabla u) + \int_{\{u^{-t}\geq u\}} f(u, \nabla u) \\
&= \int_{\{u<u^{t_*}\}} f(u, \nabla u) + \int_{\{u^{-t_*}\geq u\}} f(u, \nabla u) \\
&\leq 2\mathcal{E}(u \wedge u^{t_*}) = 0.
\end{aligned}$$

□

For the lower semicontinuity at 0, we need extra assumptions:

Proposition 2.2. *Assume that $f : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ satisfies (H1), (H2), (H4) and let H_f as defined in (1.2) (without dependence on x). Either $f(0, 0) > 0$ and H_f is identically $+\infty$ on $[0, +\infty)$, or $f(0, 0) = H_f(0) = 0$, so that H_f is in any case concave non-decreasing and lower semicontinuous on $[0, +\infty)$.*

Proof. Since $H_f(0) = \mathcal{E}(0) = f(0,0) \times (+\infty)$, in view of Theorem 2.1 it suffices to prove that H_f is lower semicontinuous at 0, thus that $\liminf_{n\to\infty} \mathcal{E}(u_n) \geq \mathcal{E}(0)$ whenever $(u_n)_{n\in\mathbb{N}}$ is a sequence of maps in $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}_+)$ converging to 0 in $L^1(\mathbb{R}^N)$. Take such a sequence and assume w.l.o.g. that $\mathcal{E}(u_n)$ is bounded. By (H4), this implies that $(u_n)_{n\in\mathbb{N}}$ is bounded in $W^{1,p}_{\mathrm{loc}}$ for some $p \in (1,+\infty)$; hence, up to extraction, we can assume that u_n converges weakly as $n \to \infty$ to some function in $W^{1,p}_{\mathrm{loc}}$ which, by L^1 convergence, must be identically 0. By lower semicontinuity of integral functionals (see [But89, Theorem 4.1.1]), we have $\liminf_{n\to\infty} \mathcal{E}(u_n) \geq \mathcal{E}(0)$. □

2.2 H-transform and H-mass

Definition 2.3. We that $H : \mathbb{R}^N \times [0,+\infty) \to [0,+\infty]$ is mass-subadditive if for every $x \in \mathbb{R}^N$ and $m_1, m_2 \in [0,+\infty)$, one has $H(x, m_1 + m_2) \leq H(x, m_1) + H(x, m_2)$.

We start with an easy lemma:

Lemma 2.4. *If $H : \mathbb{R}^N \times [0,+\infty) \to [0,+\infty]$ is mass-subadditive and admits a slope at the origin, defined for each $x \in \mathbb{R}^N$ by*

$$H'(x,0) \coloneqq \lim_{m\to 0^+} \frac{H(x,m)}{m} \in [0,+\infty], \tag{2.5}$$

then we also have

$$H'(x,0) = \sup_{m>0} \frac{H(x,m)}{m}.$$

Proof. Let $m > 0$. By subadditivity, we have for every $n \in \mathbb{N}$,

$$\frac{H(x,m)}{m} \leq \frac{nH(x,\frac{m}{n})}{m} = \frac{H(x,\frac{m}{n})}{\frac{m}{n}}.$$

In the limit $n \to \infty$, we obtain $\frac{H(x,m)}{m} \leq H'(x,0)$. Since this is true for every $m > 0$, we have $\sup_{m>0} \frac{H(x,m)}{m} \leq H'(x,0)$. The reverse inequality is obvious. □

Definition 2.5. Let $H : \mathbb{R}^N \times [0,+\infty) \to [0,+\infty]$ be a mass-subadditive function admitting a slope at the origin, as defined in (2.5). We define the *H-transform* of a positive Borel measure $u \in \mathcal{M}_+(\mathbb{R}^N)$ as:

$$H(u) \coloneqq \sum_{i\in I} H(x_i, m_i)\delta_{x_i} + H'(\cdot,0)u^d,$$

where $u = u^a + u^d$ is the decomposition of u into its atomic part $u^a = \sum_{i\in I} m_i \delta_{x_i}$, where $m_i = u(\{x_i\})$ for every $i \in I \subseteq \mathbb{N}$, and its diffuse (or non-atomic) part u^d.

The *H-mass* of u is then defined by:

$$\mathbf{M}^H(u) \coloneqq \|H(u)\| = \sum_{i\in I} H(x_i, m_i) + \int_{\mathbb{R}^N} H'(x,0)\, \mathrm{d}u^d(x).$$

$\mathbf{M}^H(u)$ is a natural spatially non-homogeneous extension (depending on the position x) of the H-mass of k-dimensional flat currents[1] from Geometric Measure Theory, introduced by [Fle66] (see also the more recent works [DH03; Col+17]).

From [BB93], we have the following result[2]:

Proposition 2.6 ([BB93, Theorem 2.4]). *If $H : \mathbb{R}^N \times [0, +\infty) \to [0, +\infty]$ is lower semicontinuous, mass-subadditive and has a slope at the origin, then $\mathbf{M}^H$ is sequentially l.s.c. on $\mathcal{M}_+(\mathbb{R}^N)$ for the weak topology.*

From the same work, in particular from [BB93, Theorem 3.2], it can be deduced that $\mathbf{M}^H$ is the relaxation for the weak topology of the functional

$$\mathbf{M}^H_{\mathrm{atom}}(u) = \begin{cases} \sum_{i=1}^k H(x_i, m_i) & \text{if } u = \sum_{i=1}^k m_i \delta_{x_i} \text{ with } k \in \mathbb{N}^*,\ m_i = u(\{x_i\}) > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

We need a slightly different result, namely that for any function $H : \mathbb{R}^N \times [0, +\infty) \to [0, +\infty]$ which is mass-subadditive, has a slope at the origin, the relaxation of $\mathbf{M}^H_{\mathrm{atom}}$ for the *narrow* sequential convergence is $\mathbf{M}^{H_{\mathrm{lsc}}}$, where H_{lsc} is the lower semicontinuous envelope of H, which can be expressed as

$$\begin{aligned} H_{\mathrm{lsc}}(x, m) &= \sup\{G(x, m) \ : \ G \leq H \text{ and } G \text{ is lower semicontinuous}\} \\ &= \inf\big\{ \liminf_{n\to\infty} H(x_n, m_n) \ : \ (x_n, m_n)_{n\in\mathbb{N}} \to (x, m),\ x_n \in \mathbb{R}^N,\ m_n \geq 0 \big\}. \end{aligned} \tag{2.6}$$

It is easy to see that H_{lsc} is still mass-subadditive, has a slope at 0 (the same as H), and $H_{\mathrm{lsc}}(\cdot, 0) \equiv 0$.

Proposition 2.7. *For any mass-subadditive function $H : \mathbb{R}^N \times [0, +\infty) \to [0, +\infty]$ which admits a slope at the origin and such that $H(\cdot, 0) \equiv 0$, the sequentially lower semicontinuous envelope of $\mathbf{M}^H_{\mathrm{atom}}$ in the narrow topology of $\mathcal{M}_+(\mathbb{R}^N)$ is given by $\mathbf{M}^{H_{\mathrm{lsc}}}$, namely we have:*

$$\mathbf{M}^{H_{\mathrm{lsc}}} = \sup\Big\{F \ : \ F \leq \mathbf{M}^H_{\mathrm{atom}},\ F \text{ sequentially narrowly l.s.c. on } \mathcal{M}_+(\mathbb{R}^N)\Big\}. \tag{2.7}$$

Note that, unlike the lower semicontinuity, the mass-subadditivity of H is not a necessary condition for the lower semicontinuity of $\mathbf{M}^H$. Indeed, $\mathbf{M}^H$ is lower semicontinuous if for instance $H(x, m) = +\infty$ when $x \neq 0$ and $H(0, \cdot)$ is any lower semicontinuous function, not necessarily subadditive. Nevertheless the mass-subadditivity would be necessary if H did not depend on x.

Proof of Proposition 2.7. Since H_{lsc} is lower semicontinuous and mass-subadditive, we know from Proposition 2.6 that $\mathbf{M}^{H_{\mathrm{lsc}}}$ is lower semicontinuous in the weak topology hence also in the narrow topology of $\mathcal{M}_+(\mathbb{R}^N)$. Since $\mathbf{M}^{H_{\mathrm{lsc}}} \leq \mathbf{M}^H_{\mathrm{atom}}$, we deduce that $\mathbf{M}^{H_{\mathrm{lsc}}}$ is lower or equal than the right hand side in (2.7).

[1] In the case $k = 0$, since signed measures are merely 0-currents with finite mass.

[2] In the notations of this paper, we take $\mu = 0$ and $f(x, s) = |s|^2$; we have $\varphi_{f,\mu}(x, 0) = 0$ and $\varphi_{f,\mu}(x, s) = +\infty$ if $s \neq 0$.

In order to prove the opposite inequality, we take a functional $F : \mathcal{M}_+(\mathbb{R}^N) \to \mathbb{R}_+$ such that $F \le \mathbf{M}^H_{\mathrm{atom}}$ and F is sequentially lower semicontinuous for the narrow convergence. We shall see that $F \le \mathbf{M}^{H_{\mathrm{lsc}}}$.

We first prove that $F \le \mathbf{M}^{H_{\mathrm{lsc}}}_{\mathrm{atom}}$. For this, we let $u = \sum_{i=1}^k m_i \delta_{x_i}$ be a finitely atomic positive measure and we let $u_n \coloneqq \sum_{i=1}^k m_{i,n} \delta_{x_{i,n}}$ where for each $i \in \{1, \dots, k\}$, $(x_{i,n})_{n\in\mathbb{N}}$ is a sequence of points converging to x_i and $m_{i,n}$ is a sequence of non-negative numbers converging to m_i such that $H_{\mathrm{lsc}}(x_i, m_i) = \lim_{n\to\infty} H(x_{i,n}, m_{i,n})$. Then $(u_n)_{n\in\mathbb{N}}$ converges narrowly to u and, by lower semicontinuity,

$$F(u) \le \liminf_{n\to\infty} F(u_n) \le \liminf_{n\to\infty} \mathbf{M}^H_{\mathrm{atom}}(u_n) = \lim_{n\to\infty} \sum_{i=1}^k H(x_{i,n}, m_{i,n}) = \sum_{i=1}^k H_{\mathrm{lsc}}(x_i, m_i),$$

so that $F(u) \le \mathbf{M}^{H_{\mathrm{lsc}}}_{\mathrm{atom}}(u)$ as wanted.

We now prove that $F \le \mathbf{M}^{H_{\mathrm{lsc}}}$. Let $u \in \mathcal{M}_+(\mathbb{R}^N)$ and let $u = u^a + u^d$ be the decomposition of u into its atomic part $u^a = \sum_{i=1}^k m_i \delta_{x_i}$, with $k \in \mathbb{N} \cup \{+\infty\}$ (here, $k = 0$ if there is no atom), and its diffuse part u^d. We then discretize u^d by taking $n \in \mathbb{N}$, a partition $(Q_i^n)_{i\in\{1,\dots,(n2^n)^N\}}$ of $[-n, n)^N$ by means of cubes of the form $Q_i^n = c_i^n + 2^{-n}[-1, 1)^N$ with $c_i^n \in \mathbb{R}^N$, and we define

$$u_n \coloneqq \sum_{i=1}^{n\wedge k} m_i \delta_{x_i} + \sum_{i=1}^{(n2^n)^N} u^d(Q_i^n) \delta_{x_i^n},$$

where for each $i \in \{1, \dots, (n2^n)^N\}$, $x_i^n \in \bar{Q}_i^n$ is some point such that

$$H'_{\mathrm{lsc}}(x_i^n, 0) = \inf_{x\in\bar{Q}_i^n} H'_{\mathrm{lsc}}(x, 0). \tag{2.8}$$

Such an x_i^n exists since $\bar{Q}_i^n$ is compact and $x \mapsto H'_{\mathrm{lsc}}(x, 0)$ is lower semicontinuous as a supremum of lower semicontinuous functions by Lemma 2.4.

The sequence $(u_n)_{n\in\mathbb{N}}$ converges narrowly to u. We deduce from the lower semicontinuity of the functional F, from the inequality $F(u) \le \mathbf{M}^{H_{\mathrm{lsc}}}_{\mathrm{atom}}(u)$, and from lemma 2.4 and (2.8), together with monotone convergence, that

$$\begin{aligned}
F(u) &\le \liminf_{n\to\infty} \sum_{i=1}^{n\wedge k} H_{\mathrm{lsc}}(x_i, m_i) + \sum_{i=1}^{(n2^n)^N} H_{\mathrm{lsc}}(x_i^n, u^d(Q_i^n)) \\
&\le \sum_{i=1}^{k} H_{\mathrm{lsc}}(x_i, m_i) + \liminf_{n\to\infty} \sum_{i=1}^{(n2^n)^N} H'_{\mathrm{lsc}}(x_i^n, 0) u^d(Q_i^n) \\
&\le \sum_{i=1}^{k} H_{\mathrm{lsc}}(x_i, m_i) + \liminf_{n\to\infty} \sum_{i=1}^{(n2^n)^N} \int_{Q_i^n} H'_{\mathrm{lsc}}(x, 0)\, \mathrm{d}u^d \\
&= \sum_{i=1}^{k} H_{\mathrm{lsc}}(x_i, m_i) + \int_{\mathbb{R}^N} H'_{\mathrm{lsc}}(x, 0)\, \mathrm{d}u^d = \mathbf{M}^{H_{\mathrm{lsc}}}(u). \qquad \square
\end{aligned}$$

2.3 Slope at the origin of the minimal cost function

Proposition 2.8. *Let $f : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ be lower semicontinuous with $N \geq 2$. For every function $\rho \in \mathcal{C}((0,1],(0,+\infty))$ such that*

$$\int_0^1 \Big(\int_y^1 \frac{\mathrm{d}t}{\rho(t)}\Big)^N \mathrm{d}y < +\infty, \tag{2.9}$$

the function H_f defined in (1.2) (without dependence on x) satisfies

$$\lim_{m\to 0^+} \frac{H_f(m)}{m} \leq \limsup_{u\to 0^+} \sup_{\xi\in\mathbb{S}^{N-1}} \frac{f(u,\rho(u)\xi)}{u}. \tag{2.10}$$

Proof. Notice that when $f(0,0) > 0$, by lower semicontinuity of f,

$$\limsup_{u\to 0^+} \sup_{\xi\in\mathbb{S}^{N-1}} \frac{f(u,\rho(u)\xi)}{u} \geq \liminf_{(u,\xi)\to(0^+,0)} \frac{f(u,\xi)}{u} = +\infty,$$

hence (2.10) is true. Assume now that $f(0,0) = 0$, let $\rho \in \mathcal{C}((0,1],(0,+\infty))$ be as in (2.9), and let

$$F(y) = \int_y^1 \frac{\mathrm{d}t}{\rho(t)} \in [0,+\infty], \quad y \geq 0.$$

The function F is decreasing, and belongs to $\mathcal{C}^1((0,1])$ and $L^N((0,1])$ by assumption. We now consider the solution of the ODE $v_\varepsilon' = -\rho(v_\varepsilon)$, with $v_\varepsilon(0) = \varepsilon$, given by

$$v_\varepsilon(r) = \begin{cases} F^{-1}(F(\varepsilon)+r), & \text{if } 0 \leq r < F(0) - F(\varepsilon), \\ 0 & \text{if } r \geq F(0) - F(\varepsilon). \end{cases}$$

Notice that $v_\varepsilon \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$ because it is nonincreasing and bounded, hence it has finite total variation, and it is of class $\mathcal{C}^1$ except possibly at $r_\varepsilon := F(0) - F(\varepsilon)$, where it has no jump. As a consequence the radial profile defined by $u_\varepsilon(x) := v_\varepsilon(|x|)$ belongs to $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$ and we compute, using the change of variables $s = v_\varepsilon(r)$ (i.e. $r = F(s) - F(\varepsilon)$) and an integration by parts combined with monotone convergence.

$$\begin{aligned} m_\varepsilon := \int_{\mathbb{R}^N} u_\varepsilon &= |\mathbb{S}^{N-1}| \int_0^\infty v_\varepsilon(r) r^{N-1} \,\mathrm{d}r \\ &= -|\mathbb{S}^{N-1}| \int_0^\varepsilon s(F(s)-F(\varepsilon))^{N-1} F'(s) \,\mathrm{d}s \\ &= |\mathbb{S}^{N-1}| \lim_{t\downarrow 0} \left(\int_t^\varepsilon \frac{(F(s)-F(\varepsilon))^N}{N} \,\mathrm{d}s - \left[s\frac{(F(s)-F(\varepsilon))^N}{N} \right]_t^\varepsilon \right) \\ &= |\mathbb{S}^{N-1}| \int_0^\varepsilon \frac{(F(s)-F(\varepsilon))^N}{N} \,\mathrm{d}s \xrightarrow[\varepsilon\to 0]{} 0. \end{aligned}$$

The equality on the last line holds because $\lim_{t\to 0^+} \int_t^\varepsilon (F-F(\varepsilon))^N < +\infty$ (since $F \in L^N((0,1])$), hence $\lim_{t\to 0} t(F(t)-F(\varepsilon))^N$ exists by existence of the limit in the previous line, and it must be zero (again, because $F \in L^N((0,1])$).

Moreover, since $\sup_{[0,+\infty)} v_\varepsilon = \varepsilon$,

$$\mathcal{E}(u_\varepsilon) = \int_0^\infty \int_{\mathbb{S}^{N-1}} f(v_\varepsilon(r), v_\varepsilon'(r)\xi) r^{N-1} \,\mathrm{d}\mathcal{H}^{N-1}(\xi) \,\mathrm{d}r \le m_\varepsilon \sup_{u\le\varepsilon,\, |\xi|=1} \frac{f(u,\rho(u)\xi)}{u}.$$

By assumption, we deduce that

$$\limsup_{m\to 0^+} \frac{H(m)}{m} \le \limsup_{\varepsilon\to 0^+} \frac{\mathcal{E}(u_\varepsilon)}{m_\varepsilon} \le \limsup_{u\to 0^+} \sup_{\xi\in\mathbb{S}^{N-1}} \frac{f(u,\rho(u)\xi)}{u}. \qquad \square$$

In dimension $N = 1$, we need no other assumption than $H < +\infty$, as stated below.

Proposition 2.9. *Let $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+ \cup \{+\infty\}$ be Borel measurable. The function H defined by (2.1) (with $N = 1$) is either identically infinite on $(0, +\infty)$, or it satisfies (2.10) with $\rho \equiv 0$.*

Proof. One can assume that there exists $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}_+)$ with $0 < \int_{\mathbb{R}} u < +\infty$ and $\mathcal{E}(u) < +\infty$. In particular, up to changing the value of u on a negligible set, u is continuous on $\mathbb{R}$. Let $\varepsilon \in (0, \sup_{\mathbb{R}} u)$, set $A_\varepsilon := \{x : u(x) = \varepsilon\}$ which is non-empty by the intermediate value theorem and integrability of u, and define

$$a_\varepsilon = \begin{cases} \inf A_\varepsilon & \text{if } \inf A_\varepsilon > -\infty, \\ \text{any point in } (-\infty, -\varepsilon^{-1}) \cap A_\varepsilon & \text{otherwise,} \end{cases}$$
$$b_\varepsilon = \begin{cases} \sup A_\varepsilon & \text{if } \sup A_\varepsilon < +\infty, \\ \text{any point in } (\varepsilon^{-1}, +\infty) \cap A_\varepsilon & \text{otherwise.} \end{cases}$$

By continuity and integrability of u, $u(a_\varepsilon) = u(b_\varepsilon) = \varepsilon$ and $u < \varepsilon$ on $\mathbb{R}\backslash[a_\varepsilon, b_\varepsilon]$. Moreover $a_\varepsilon, b_\varepsilon$ converge to points $-\infty \le a \le b \le +\infty$, hence $u = 0$ on $\mathbb{R} \setminus (a, b)$ and by dominated convergence, since $\nabla u = 0$ a.e. on $\{u = 0\}$,

$$+\infty > \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}\backslash[a_\varepsilon,b_\varepsilon]} u + f(u, \nabla u) = f(0, 0)\, \mathcal{L}(\mathbb{R} \setminus (a, b)).$$

Notice that this limit is necessary zero. Let $m > 0$. If ε is small enough, then $\int_{\mathbb{R}\backslash[a_\varepsilon,b_\varepsilon]} u < m$ so that we can take $R_\varepsilon > 0$ such that $\varepsilon R_\varepsilon = m - \int_{\mathbb{R}\backslash[a_\varepsilon,b_\varepsilon]} u$. We then define

$$u_\varepsilon(x) = \begin{cases} u(x) & \text{if } x \le a_\varepsilon, \\ \varepsilon & \text{if } a_\varepsilon < x < a_\varepsilon + R_\varepsilon, \\ u(b_\varepsilon + x - (a_\varepsilon + R_\varepsilon)) & \text{if } x \ge a_\varepsilon + R_\varepsilon, \end{cases}$$

so that $\int_{\mathbb{R}} v_\varepsilon = m$. Moreover,

$$\mathcal{E}(v_\varepsilon) = \mathcal{E}(u, \mathbb{R} \setminus [a_\varepsilon, b_\varepsilon]) + R_\varepsilon f(\varepsilon, 0).$$

Hence, as $R_\varepsilon = \frac{m+o(1)}{\varepsilon}$ as $\varepsilon \to 0$,

$$H(m) \le \limsup_{\varepsilon\to 0^+} \mathcal{E}(v_\varepsilon) = m \limsup_{\varepsilon\to 0^+} \frac{f(\varepsilon, 0)}{\varepsilon}. \qquad \square$$

3 Lower bound for the energy and existence of optimal profiles

Our main tool to localize the energy and obtain a lower bound relies on a profile decomposition for bounded sequences of positive measures, which is reminiscent of the concentration compactness principle of P.-L. Lions. This differs from classical strategies to localize the energy which are based on suitable cut-offs. Naturally, this concentration compactness result also provides a criterion for the existence of optimal profiles in (1.2).

3.1 Profile decomposition by concentration compactness

We prove a profile decomposition theorem for bounded sequences of positive measures over $\mathbb{R}^N$, which is essentially equivalent to [Mar14, Theorem 1.5] in the Euclidean case. We have added an extra information on mass conservation that will be useful, and provide a self-contained simple proof. We start with a definition.

Definition 3.1. A sequence of positive measures $(\mu_n)_{n\in\mathbb{N}} \in \mathcal{M}_+(\mathbb{R}^N)$ is *vanishing* if

$$\sup_{x\in\mathbb{R}^N} \mu_n(B_1(x)) \xrightarrow[n\to\infty]{} 0.$$

Any bounded sequence of positive measures over $\mathbb{R}^N$ may be decomposed (up to subsequence) into a countable collection of narrowly converging "bubbles" and a vanishing part, accounting for the total mass of the sequence, as stated in the following theorem.

Theorem 3.2. *For every bounded sequence $(\mu_n)_{n\in\mathbb{N}}$ of positive Borel measures on $\mathbb{R}^N$, there exists a subsequence $(\mu_n)_{n\in\sigma(\mathbb{N})}$, $\sigma \in \Sigma$, a non-decreasing sequence of integers $(k_n)_{n\in\sigma(\mathbb{N})}$ converging to some $k \in \mathbb{N}\cup\{+\infty\}$, a sequence of non-trivial positive Borel measures $(\mu^i)_{0\le i<k}$, and for every $n \in \sigma(\mathbb{N})$, a collection of balls $(B_n^i)_{0\le i<k_n}$ centered at points of $\operatorname{supp}\mu_n$ such that, writing for all $n \in \sigma(\mathbb{N})$,*

$$\mu_n = \mu_n^b + \mu_n^v, \quad \textit{where } \mu_n^b = \sum_{0\le i<k_n} \mu_n \llcorner B_n^i, \tag{3.1}$$

(A) bubbles emerge: $(c_{B_n^i}\mu_n)_{n\in\sigma(\mathbb{N})} \xrightarrow[n\to\infty]{\mathcal{C}_b'} \mu^i$ for every $i < k$,[3]

(B) bubbles split: $\min_{0\le i<j<k_n} \operatorname{dist}(B_n^i, B_n^j) \xrightarrow[n\to\infty]{} +\infty$,

(C) bubbles diverge: $\min_{0\le i<k_n} \operatorname{diam}(B_n^i) \xrightarrow[n\to\infty]{} +\infty$,

(D) the bubbling mass is conserved: $\|\mu_n^b\| \xrightarrow[\ell\to\infty]{} \sum_{0\le i<k}\|\mu^i\|$,

(E) the remaining part is vanishing: $\sup_{x\in\mathbb{R}^N} \mu_n^v(B_1(x)) \xrightarrow[n\to\infty]{} 0$.

[3]Recall that $c_B\mu = (x \mapsto x-y)_\sharp(\mu\llcorner B)$ if $B = B_r(y)$ and $\mu \in \mathcal{M}(\mathbb{R}^N)$.

Before proving Theorem 3.2, we introduce the "bubbling" function of a sequence of finite *signed* measures $(\mu_n)_{n\in\mathbb{N}}$:

$$m((\mu_n)_{n\in\mathbb{N}}) := \sup\Big\{\|\mu\| \ : \ (\tau_{-x_{\sigma(\ell)}}\mu_{\sigma(\ell)})_{\ell\in\mathbb{N}} \xrightarrow{\mathcal{C}_0'} \mu,\ \sigma\in\Sigma,\ x_{\sigma(\ell)}\in\mathbb{R}^N\ (\forall\ell)\Big\}. \tag{3.2}$$

Although we will use this function on signed measures, we will start from a sequence of positive measures and use the following characterization of vanishing sequences, which holds only in the case of positive measures:

Lemma 3.3. *A sequence $(\mu_n)_{n\in\mathbb{N}}$ of finite positive measures is vanishing if and only if $m((\mu_n)_{n\in\mathbb{N}})=0$.*

Proof. Assume that $(\mu_n)_{n\in\mathbb{N}}$ is vanishing and that $(\tau_{-x_{\sigma(\ell)}}\mu_{\sigma(\ell)})_{\ell\in\mathbb{N}} \xrightarrow{\mathcal{C}_0'} \mu$ for some $\sigma\in\Sigma$ and some sequence of points $(x_{\sigma(\ell)})_{\ell\in\mathbb{N}}$. Then, for every $x\in\mathbb{R}^N$,

$$\mu(B_1(x)) \le \liminf_{\ell\to\infty} \tau_{-x_{\sigma(\ell)}}\mu_{\sigma(\ell)}(B_1(x)) = \liminf_{\ell\to\infty} \mu_{\sigma(\ell)}(B_1(x+x_{\sigma(\ell)})) = 0,$$

i.e. $\mu=0$ and thus $m((\mu_\ell)_{\ell\in\mathbb{N}})=0$.

Conversely, if $(\mu_n)_{n\in\mathbb{N}}$ is not vanishing, then there exists $\varepsilon>0$, $\sigma\in\Sigma$ a sequence of points $(x_n)_{n\in\sigma(\mathbb{N})}$ in $\mathbb{R}^N$ such that $\mu_n(B_1(x_n))\ge\varepsilon$ for every $n\in\sigma(\mathbb{N})$. Up to further extraction, one can assume that $(\tau_{-x_{\sigma(\ell)}}\mu_{\sigma(\ell)})_{\ell\in\mathbb{N}} \xrightarrow{\mathcal{C}_0'} \mu\in\mathcal{M}(\mathbb{R}^N)$. We have

$$\mu(\bar{B}_1(0)) \ge \limsup_{\ell\to\infty} \tau_{-x_{\sigma(\ell)}}\mu_{\sigma(\ell)}(\bar{B}_1(0)) = \limsup_{\ell\to\infty} \mu_{\sigma(\ell)}(\bar{B}_1(x_{\sigma(\ell)})) \ge \varepsilon > 0,$$

which entails $m((\mu_\ell)_{\ell\in\mathbb{N}})\ge\varepsilon>0$. □

Proof of Theorem 3.2. If $(\mu_n)_{n\in\mathbb{N}}$ is vanishing, then we take $\sigma=\mathrm{Id}$ and $k=0$, so that $\mu_{\sigma(\ell)}=\mu_\ell=\mu_\ell^v$, (A) to (D) are empty statements and (E) is satisfied since $(\mu_n)_{n\in\mathbb{N}}$ is vanishing. Assume on the contrary that $(\mu_n)_{n\in\mathbb{N}}$ is not vanishing. We shall construct the bubbles by induction and prove their properties in several steps.

Step 1: construction of bubbles centers. At first step (step 0), since $m((\mu_n)_{n\in\mathbb{N}})>0$, there exists $\sigma_0\in\Sigma$ and a sequence of points $(x_n^0)_{n\in\sigma_0(\mathbb{N})}$, such that

$$(\tau_{-x_n^0}\mu_n)_{n\in\sigma_0(\mathbb{N})} \xrightarrow{\mathcal{C}_0'} \mu^0\in\mathcal{M}(\mathbb{R}^N) \quad \text{with} \quad \|\mu^0\|\ge\frac{1}{2}m((\mu_n)_{n\in\mathbb{N}}). \tag{3.3}$$

We then set $\mu_n^0 := \mu_n - \tau_{x_n^0}\mu^0$ and we continue by induction, starting from the sequence $(\mu_n^0)_{n\in\sigma_0(\mathbb{N})}$. More precisely, assume that for a fixed step $k-1\in\mathbb{N}$, for every $i\in\mathbb{N}$ with $0\le i\le k-1$, we have built $\mu^i\in\mathcal{M}(\mathbb{R}^N)$, $\sigma_i\in\Sigma$, points $(x_n^i)_{n\in\sigma_i(\mathbb{N})}$ and sequences $(\mu_n^i)_{n\in\sigma_i(\mathbb{N})}\in\mathcal{M}(\mathbb{R}^N)$ such that for every i,

$$\sigma_i \preceq \sigma_{i-1}, \tag{3.4}$$

$$\mu_n^i = \mu_n - \sum_{0\le j\le i}\tau_{x_n^j}\mu^j, \quad (\forall n\in\sigma_i(\mathbb{N})), \tag{3.5}$$

$$(\tau_{-x_n^i}\mu_n^{i-1})_{n\in\sigma_i(\mathbb{N})} \xrightarrow{\mathcal{C}_0'} \mu^i, \tag{3.6}$$

$$\|\mu^i\| \ge \frac{1}{2}m((\mu_n^i)_{n\in\sigma_i(\mathbb{N})}) > 0, \tag{3.7}$$

where $\sigma_{-1} := \mathrm{Id}, (\mu_n^{-1}) := (\mu_n)$. If $m((\mu_n^{k-1})_{n\in\sigma_{k-1}(\mathbb{N})}) = 0$, we stop; otherwise, we proceed to the next step k to build $\sigma_k, \mu^k, (x_n^k)_{n\in\sigma_k(\mathbb{N})}, (\mu_n^k)$ as we did at step $k = 0$, starting with $(\mu_n^{k-1})_{n\in\sigma_{k-1}(\mathbb{N})}$. Either the induction stops at some step $k-1 \in \mathbb{N}$ for which $m((\mu_n^{k-1})_{n\in\sigma_{k-1}(\mathbb{N})}) = 0$ or the previous objects are defined for every $i \in \mathbb{N}$, in which case we let $k := +\infty$.

Step 2: splitting of bubbles centers. We prove that

$$\lim_{\sigma_i(\mathbb{N})\ni n\to\infty} \mathrm{dist}(x_n^i, x_n^j) = +\infty \quad \text{for every } i, j \in \mathbb{N} \text{ with } 0 \le j < i < k. \tag{3.8}$$

Indeed, assume by contradiction that there is a first index $i < k$ such that for some $j_0 < i$, $(\mathrm{dist}(x_n^i, x_n^{j_0}))_{n\in\sigma_i(\mathbb{N})}$ is not divergent. In particular, there exists $\sigma \preceq \sigma_i$ such that $(x_n^i - x_n^{j_0})_{n\in\sigma(\mathbb{N})} \to x \in \mathbb{R}^N$. Moreover, $(\mathrm{dist}(x_n^i, x_n^j))_{n\in\sigma_i(\mathbb{N})} \to \infty$, for every $j < i$, $j \neq j_0$ by minimality of i and the triangle inequality $\mathrm{dist}(x_n^j, x_n^{j_0}) \le \mathrm{dist}(x_n^j, x_n^i) + \mathrm{dist}(x_n^i, x_n^{j_0})$. Notice by (3.5) that for every $n \in \sigma(\mathbb{N})$,

$$\mu_n^{i-1} = \mu_n^{j_0-1} - \tau_{x_n^{j_0}}\mu^{j_0} - \sum_{j_0<j<i} \tau_{x_n^j}\mu^j,$$

hence taking the translation $\tau_{-x_n^i}$,

$$\tau_{-x_n^i}\mu_n^{i-1} = \tau_{x_n^{j_0}-x_n^i}(\tau_{-x_n^{j_0}}\mu_n^{j_0-1} - \mu^{j_0}) - \sum_{j_0<j<i} \tau_{x_n^j - x_n^i}\mu^j,$$

and passing to the weak limit, knowing that $x_n^{j_0} - x_n^i \to -x$ and $\mathrm{dist}(x_n^j, x_n^i) \to +\infty$ for $j_0 < j < i$,

$$\mu^i = \tau_{-x}(\mu^{j_0} - \mu^{j_0}) - \sum_{j_0<j<i} 0 = 0.$$

This contradicts the fact that $(\tau_{-x_n^i}\mu_n^{i-1})_{n\in\sigma(\mathbb{N})} \xrightarrow{\mathcal{C}_0'} \mu^i \neq 0$ and proves (3.8).

Step 3: weak convergence of bubbles. From (3.6) we get

$$\tau_{-x_n^i}\mu_n^{i-1} = \tau_{-x_n^i}\mu_n - \sum_{0\le j<i} \tau_{-x_n^i + x_n^j}\mu^j, \tag{3.9}$$

and by (3.8), the sum converges weakly to 0, and so

$$(\tau_{-x_n^i}\mu_n)_{n\in\sigma_i(\mathbb{N})} \xrightarrow{\mathcal{C}_0'} \mu^i \quad \text{for every } i \in \mathbb{N} \text{ with } i < k. \tag{3.10}$$

Step 4: construction of the bubbles with mass conservation. We now construct the extraction $\sigma \in \Sigma$ that we need by induction: we set $\sigma(0) = 0$ and, assuming that $\sigma(0) < \cdots < \sigma(\ell - 1)$, with $\ell \in \mathbb{N}^*$, have been constructed, we set $\sigma(\ell) := n$ with $n \in \sigma_{\ell\wedge k-1}(\mathbb{N})$ large enough so that $n > \sigma(\ell - 1)$ and for every $i < \ell \wedge k$,

$$\mu_n(B_\ell(x_n^i)) \le \|\mu^i\| + 2^{-\ell}, \tag{3.11}$$

and

$$\min_{0\le j<i} \mathrm{dist}(x_n^i, x_n^j) \ge 4\ell. \tag{3.12}$$

Such an n exists by (3.8) and (3.10), noticing that $\mu_n(B_\ell(x_n^i)) = (\tau_{-x_n^i}\mu_n)(B_\ell)$. Then for each $n = \sigma(\ell)$, $\ell \in \mathbb{N}$, we set $k_n = \ell \wedge k$, and for each $i \in \{0, \dots, k_n - 1\}$,

$$B_n^i := B_\ell(x_n^i).$$

Finally, for every $n \in \sigma(\mathbb{N})$, we decompose μ_n as expected:

$$\mu_n = \mu_n^b + \mu_n^v, \quad \text{where } \mu_n^b = \sum_{0 \le i < k_n} \mu_n \llcorner B_n^i.$$

Let us check the four first items (A)–(D). Notice that (C) is fulfilled because $\operatorname{diam}(B_{\sigma(\ell)}^i) = \ell \to +\infty$ as $\ell \to \infty$, and (B) because of (3.12). Since for every $i < k$, $\lim_{\sigma(\mathbb{N}) \ni n \to \infty} \operatorname{diam}(B_n^i) = +\infty$ and $c_{B_n^i}\mu_n = (\tau_{-x_n^i}(\mu_n \llcorner B_n^i))$ for every $n \in \sigma_i(\mathbb{N})$, $(c_{B_n^i}\mu_n)_{n \in \sigma(\mathbb{N})}$ converges weakly to μ^i by (3.10), and together with (3.11) it implies that

$$(c_{B_n^i}\mu_n)_{n \in \sigma(\mathbb{N})} \xrightharpoonup{\mathcal{C}_b'} \mu^i,$$

i.e. (A) is satisfied. Moreover, by (3.11) again,

$$\limsup_{\ell \to \infty} \sum_{0 \le i < k_{\sigma(\ell)}} \mu_{\sigma(\ell)}(B_{\sigma(\ell)}^i) \le \sum_{0 \le i < k} \|\mu^i\| + \limsup_{\ell \to \infty} (\ell \wedge k) 2^{-\ell} = \sum_{0 \le i < k} \|\mu^i\|,$$

and since $k_n \to k$, by Fatou's lemma we have,

$$\sum_{0 \le i < k} \|\mu^i\| \le \liminf_{\ell \to \infty} \sum_{0 \le i < k_{\sigma(\ell)}} \mu_{\sigma(\ell)}(B_{\sigma(\ell)}^i),$$

which proves (D) because $\sum_{0 \le i < k_{\sigma(\ell)}} \mu_{\sigma(\ell)}(B_{\sigma(\ell)}^i) = \|\mu_{\sigma(\ell)}^b\|$.

Step 5: vanishing of the remaining part, proof of (E). By Lemma 3.3, it suffices to prove that $m((\mu_n^v)_{n \in \sigma(\mathbb{N})}) = 0$. We claim that:

$$m((\mu_n^v)_{n \in \sigma(\mathbb{N})}) \le m((\mu_n^i)_{n \in \sigma_i(\mathbb{N})}), \quad \text{for every } i \in \mathbb{N} \text{ with } i < k, \tag{3.13}$$

which concludes since $m((\mu_n^k)_{n \in \sigma_{k-1}(\mathbb{N})}) = 0$ if $k < \infty$, and $m((\mu_n^i))_{n \in \sigma_i(\mathbb{N})}) \to 0$ as $i \to \infty$ if $k = \infty$. Indeed, if $k = \infty$, we have by (3.7) and (D),

$$\frac{1}{2} \sum_{i \in \mathbb{N}} m((\mu_n^i)_{n \in \sigma_i(\mathbb{N})}) \le \sum_{i \in \mathbb{N}} \|\mu^i\| = \lim_{\ell \to \infty} \|\mu_{\sigma(\ell)}^b\| \le \liminf_{\ell \to \infty} \|\mu_{\sigma(\ell)}\| < \infty.$$

Let us show (3.13). Let $\bar{\sigma} \preceq \sigma$ and $(x_n)_{n \in \bar{\sigma}(\mathbb{N})}$ be a sequence of points such that

$$(\tau_{-x_n}\mu_n^v)_{n \in \bar{\sigma}(\mathbb{N})} \xrightharpoonup{\mathcal{C}_0'} \mu \in \mathcal{M}(\mathbb{R}^N).$$

We need to prove that $\|\mu\| \le m((\mu_n^i)_{n \in \sigma_i(\mathbb{N})})$ for every $i < k$. Assume without loss of generality that $\|\mu\| > 0$. Then for every $i < k$,

$$(\operatorname{dist}(x_n, x_n^i))_{n \in \bar{\sigma}(\mathbb{N})} \to \infty. \tag{3.14}$$

Otherwise, up to subsequence, $(\mathrm{dist}(x_n, x_n^i))_n$ would be bounded by some constant M, and for every $r > 0$,

$$(\tau_{-x_n}\mu_n^v)(B_r) \le \mu_n^v(B_{r+M}(x_n^i)) \xrightarrow[n\to\infty]{} 0,$$

because μ_n^v is supported on $\mathbb{R}^N \setminus \cup_{0\le i<k_n} B_n^i$ and $B_{r+M}(x_n^i) \subseteq B_n^i$ for n large enough by (E). Hence μ would be 0, a contradiction. Up to further extraction, one can assume that $(\tau_{-x_n}\mu_n)_{n\in\bar{\sigma}(\mathbb{N})}$ converges weakly to a measure $\bar{\mu} \in \mathcal{M}(\mathbb{R}^N)$. Since $\mu_n^v \le \mu_n$, we have $\mu \le \bar{\mu}$. Moreover by (3.5), for every $i < k$ and $n \in \bar{\sigma}(\mathbb{N})$ large enough,

$$\tau_{-x_n}\mu_n^i = \tau_{-x_n}\mu_n - \sum_{0\le j\le i} \tau_{x_n^j - x_n}\mu^j,$$

and because of (3.14) the sum converges weakly to 0, so that $\tau_{-x_n}\mu_n^i \xrightarrow{\mathcal{C}_0'} \bar{\mu}$, and consequently,

$$\|\mu\| \le \|\bar{\mu}\| \le m((\mu_n^i)_{n\in\sigma_i(\mathbb{N})}),$$

which is what had to be proved.

Step 6: re-centering of the bubbles at points of $\operatorname{supp}\mu_n$*.* By (3.10), $(\tau_{-x_n^i}\mu_n)_{n\in\sigma(\mathbb{N})}$ converges weakly to the non-trivial measure μ_i for every $i < k$, thus

$$R_i/2 \coloneqq \limsup_{\sigma(\mathbb{N})\ni n\to+\infty} \mathrm{dist}(\operatorname{supp}\mu_n, x_n^i) < +\infty. \tag{3.15}$$

Therefore, for every n large enough, there is a point $\tilde{x}_n^i$ such that $|x_n^i - \tilde{x}_n^i| < R_i$ and $\tilde{x}_n^i \in \operatorname{supp}\mu_n$. After a further extraction, one may assume that for every i, $|x_n^i - \tilde{x}_n^i| < R_i < r_i^n$ with $\operatorname{diam} B_n^i = 2r_n^i$ for every n, and $(x_n^i - \tilde{x}_n^i)_{n\in\sigma(\mathbb{N})}$ converges to some $p_i \in \mathbb{R}^N$. Finally, we set $\tilde{r}_i^n \coloneqq r_i^n - R_i$ and $\tilde{B}_n^i \coloneqq B(\tilde{x}_n^i, \tilde{r}_i^n) \subseteq B_n^i$. After replacing the balls B_n^i by $\tilde{B}_n^i$, (B) and (C) are satisfied by definition. Notice that $(\tau_{-\tilde{x}_n^i}\mu_n)_{n\in\sigma(\mathbb{N})}$ converges weakly to $\tilde{\mu}^i \coloneqq \tau_{p_i}\mu^i$ with $\|\tilde{\mu}^i\| = \|\mu^i\|$, and $\limsup_n \|c_{B_n^i}\mu_n\| = \limsup_n \mu_n(\tilde{B}_n^i) \le \limsup_n \mu_n(B_n^i) = \|\mu^i\|$ hence (A) holds. Besides, using Fatou's lemma,

$$\begin{aligned}\limsup_n \sum_{i<k_n} \mu_n(\tilde{B}_n^i) \le \limsup_n \sum_{i<k_n} \mu_n(B_n^i) = \sum_{i<k} \|\mu^i\| \\ \le \sum_{i<k} \liminf_n \mu_n(\tilde{B}_n^i) \le \liminf_n \sum_{i<k_n} \mu_n(\tilde{B}_n^i)\end{aligned}$$

so that $\lim_n \sum_{i<k_n} \mu_n(\tilde{B}_n^i) = \sum_i \|\mu_i\|$ and (D) is satisfied. In particular, $\lim_n \sum_{i<k_n} \mu_n(B_n^i \setminus \tilde{B}_n^i) = \lim_n \sum_{i<k_n} \mu_n(B_n^i) - \lim_n \sum_{i<k_n} \mu_n(\tilde{B}_n^i) = 0$ and (E) holds as well. □

Remark 3.4. If the sequence of families of balls $(B_n^i)_{0\le i<k_n}$ satisfies the conclusion of the theorem, i.e. (A)–(E), then it is also the case for any family of balls $(\tilde{B}_n^i)_{0\le i<k_n}$ with the same centers as those of B_n^i and with smaller but still divergent radii (i.e. satisfying (C)). It can be easily seen following the arguments at Step 6 of the proof.

3.2 Lower bound by concentration compactness

We will first establish a lower bound for the minimal energy along vanishing sequences defined on varying subsets of $\mathbb{R}^N$. We say that a sequence of Borel functions $(u_n)_{n\in\mathbb{N}}$, each defined on some open set $\Omega_n \subseteq \mathbb{R}^N$, is vanishing if the sequence of measures $(|u_n|\,\mathcal{L}^N \llcorner \Omega_n)_{n\in\mathbb{N}}$ is vanishing in the sense of Definition 3.1, namely if $\|u_n\|_{L^1_{\mathrm{uloc}}(\Omega_n)} \to 0$ as $n \to \infty$, where $L^1_{\mathrm{uloc}}(\Omega)$ is the set of uniformly locally integrable functions on the open set Ω, i.e. Borel functions u on Ω such that

$$\|u\|_{L^1_{\mathrm{uloc}}(\Omega)} := \sup_{x\in\mathbb{R}^N} \int_{\Omega\cap(x+[0,1)^N)} |u| < +\infty. \tag{3.16}$$

It will be convenient to first extend our Sobolev functions to a neighbourhood Ω_δ of Ω where for every $\delta > 0$ and every set $X \subseteq \mathbb{R}^N$, we have set

$$X_\delta := \{x \in \mathbb{R}^N \ : \ \mathrm{dist}(x, X) < \delta\}.$$

We will need to consider sufficiently regular domains for which we have an extension operator $W^{1,p} \cap L^1_{\mathrm{uloc}}(\Omega) \to W^{1,p} \cap L^1_{\mathrm{uloc}}(\Omega_\delta)$. We will only apply it to domains with smooth boundary, in which case we can use a reflexion technique. Since we want quantitative estimates, we will use the notion of *reach* of a set $X \subseteq \mathbb{R}^N$ (see [Fed59]). We say that X has positive reach if there exists $\delta > 0$ such that every $x \in X_\delta$ has a unique nearest point $\pi(x)$ on X. The greatest δ for which this holds is denoted by $\mathrm{reach}(X)$ and the map $x \in X_{\mathrm{reach}(X)} \mapsto \pi(x) \in X$ is called the nearest point retraction.

Example 3.5. Assume that Ω is a perforated domain $B^0 \setminus \bigcup_{i=1}^k B^i$ where the B^i are disjoint closed balls included in some open ball B^0 (possibly $B^0 = \mathbb{R}^N$). Then,

$$\mathrm{reach}(\partial\Omega) = \inf\{\mathrm{radius}(B^i) \ : \ i = 0, \dots, k\} \cup \{\mathrm{dist}(\partial B^i, \partial B^j) \ : \ i \neq j\}.$$

By [Fed59, Theorem 4.8], we have

i) if $x, y \in X_\delta$ with $0 < \delta < \delta_0 := \mathrm{reach}(X)$, then $|\pi(x) - \pi(y)| \leq \frac{\delta_0}{\delta_0 - \delta}|x - y|$,

ii) if $x \in X$ and D_x is the intersection of $X_{\mathrm{reach}(X)}$ with the straight line crossing $\partial\Omega$ orthogonally at x, then $\pi(y) = x$ for every $y \in D_x$.

Lemma 3.6 (Extension)**.** *Let $\Omega \subseteq \mathbb{R}^N$ be an open set such that its boundary $\partial\Omega$ is $\mathcal{C}^1$ with positive reach. Then, for every $\delta \in (0, \mathrm{reach}(\partial\Omega))$, every $p \in [1, +\infty)$ and every $u \in L^1 \cap W^{1,p}(\Omega)$, there exists $\bar{u} \in L^1 \cap W^{1,p}(\Omega_\delta)$ such that $\bar{u} = u$ a.e. on Ω, and*

$$\|\bar{u}\|_{L^1(\Omega_\delta)} \leq A\|u\|_{L^1(\Omega)}, \quad \|\bar{u}\|_{L^1_{\mathrm{uloc}}(\Omega_\delta)} \leq A\|u\|_{L^1_{\mathrm{uloc}}(\Omega)}, \quad \|\nabla\bar{u}\|_{L^p(\Omega_\delta)} \leq A\|\nabla u\|_{L^p(\Omega)},$$

with a constant $A < +\infty$ depending only on N, δ and $\mathrm{reach}(\partial\Omega)$.

Proof. Let $\sigma : (\partial\Omega)_\delta \to (\partial\Omega)_\delta$ be the reflexion through $\partial\Omega$, defined by $\sigma(x) = 2\pi(x) - x$. By the properties i) and ii) of the nearest point retraction, we have that $\sigma = \sigma^{-1}$ (simply because $\pi(\sigma(x)) = \pi(x)$) and σ is L-Lipschitz with a constant $L < +\infty$ depending on δ and $\mathrm{reach}(\partial\Omega)$ only.

We define $\bar{u}$ by $\bar{u} = u$ on Ω and $\bar{u} = u \circ \sigma$ on $\Omega_\delta \setminus \Omega$[4]. This map is well defined since $\sigma(\Omega_\delta \setminus \Omega) \subseteq \Omega$. Indeed, if we had $x, \sigma(x) \in \Omega_\delta \setminus \Omega$, then the line segment $[x, \sigma(x)]$ would meet $\partial\Omega$ orthogonally at its center $\pi(x)$, and would remain out of Ω elsewhere, because otherwise there would exist a point y belonging either to $\partial\Omega \cap (x, \pi(x))$ or $\partial\Omega \cap (\pi(x), \sigma(x))$ thus contradicting the definition of $\pi(x)$. Such a situation is not possible for a $\mathcal{C}^1$ boundary.

Moreover, by the change of variable formula and the chain rule, $\bar{u}$ satisfies the desired estimates since σ is bi-Lipschitz with its Lipschitz constants controlled in terms of δ and $\mathrm{reach}(\partial\Omega)$. □

We will need a localized version of the Gagliardo–Nirenberg–Sobolev inequality in a particular case:

Lemma 3.7. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set such that $\partial\Omega$ is $\mathcal{C}^1$ with positive reach, let $p \in [1, +\infty)$, let $r \geq p(1 + \frac{1}{N})$, and assume that $r \leq \frac{pN}{N-p}$ when $p < N$. Then for every $u \in L^1 \cap W^{1,p}(\Omega)$,*

$$\|u\|_{L^r(\Omega)} \leq C\big(\|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^1(\Omega)}\big)^\alpha \|u\|^{1-\alpha}_{L^1_{\mathrm{uloc}}(\Omega)},$$

where $\alpha \in (0, 1]$ is the unique parameter such that $\frac{1}{r} = \alpha(\frac{1}{p} - \frac{1}{N}) + (1 - \alpha)$, and the constant $C < +\infty$ depends on N, r, p and $\mathrm{reach}(\partial\Omega)$.

Proof of Lemma 3.7. We let $u \in L^1 \cap W^{1,p}(\Omega)$ and we extend u to $\bar{u} \in L^1 \cap W^{1,p}(\Omega_\delta)$ as in Lemma 3.6, with $\delta \coloneqq \mathrm{reach}(\Omega)/2$. By the Gagliardo–Nirenberg–Sobolev inequality (see [Nir59]) on the hypercube $Q_\delta = [-\frac{\delta}{\sqrt{N}}, \frac{\delta}{\sqrt{N}})^N$, we have

$$\|\bar{u}\|_{L^r(Q_\delta)} \leq C\|\nabla \bar{u}\|^\alpha_{L^p(Q_\delta)} \|\bar{u}\|^{1-\alpha}_{L^1(Q_\delta)} + C\|\bar{u}\|_{L^1(Q_\delta)}.$$

We then cover Ω with the hypercubes $Q_\delta(c) = c + Q_\delta \subseteq \Omega_\delta$ centered at points c on the grid $\mathcal{C} \coloneqq \Omega \cap \delta\mathbb{Z}^N$. Since $\alpha \geq \frac{N}{N+1}$, we can check that

$$r\alpha = \frac{r-1}{1 + \frac{1}{N} - \frac{1}{p}} \geq p. \tag{3.17}$$

By superadditivity of $s \mapsto s^{\frac{r\alpha}{p}}$ and of $s \mapsto s^{r\alpha}$, we obtain

$$\begin{aligned}
\|u\|^r_{L^r(\Omega)} &\leq \sum_{c \in \mathcal{C}} \|\bar{u}\|^r_{L^r(Q_\delta(c))} \\
&\leq C' \sum_{c \in \mathcal{C}} \|\nabla \bar{u}\|^{p\frac{r\alpha}{p}}_{L^p(Q_\delta(c))} \|\bar{u}\|^{r(1-\alpha)}_{L^1(Q_\delta(c))} + C'\|\bar{u}\|^r_{L^1(Q_\delta(c))} \\
&\leq C''\|\nabla \bar{u}\|^{r\alpha}_{L^p(\Omega_\delta)} \|\bar{u}\|^{r(1-\alpha)}_{L^1_{\mathrm{uloc}}(\Omega_\delta)} + C'\|\bar{u}\|^{r\alpha}_{L^1(\Omega_\delta)} \|\bar{u}\|^{r(1-\alpha)}_{L^1_{\mathrm{uloc}}(\Omega_\delta)} \\
&\leq C'''\big(\|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^1(\Omega)}\big)^{r\alpha} \|u\|^{r(1-\alpha)}_{L^1_{\mathrm{uloc}}(\Omega)}.
\end{aligned}$$

□

[4] Note that $\bar{u}$ is not defined on $\partial\Omega$, but this set is negligible.

Proposition 3.8. *Assume that $f : \mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}_+$ satisfies (H1) and (H4) for some $p \in (1, +\infty)$. Consider a vanishing sequence $(u_n)_{n\in\mathbb{N}}$ in $W^{1,1}_{\mathrm{loc}}(\Omega_n, \mathbb{R}_+)$, where the $\Omega_n \subseteq \mathbb{R}^N$ are open sets with $\mathcal{C}^1$ boundary and such that $\inf_{n\in\mathbb{N}} \mathrm{reach}(\partial\Omega_n) > 0$, and a sequence $(\Phi_n)_{n\in\mathbb{N}}$ of Borel maps $\Phi_n : \Omega_n \to \mathbb{R}^N$ such that $\sup_{y\in\Omega_n} |\Phi_n(y) - x_0| \to 0$ as $n \to +\infty$ for some $x_0 \in \mathbb{R}^N$. If $\theta_n := \int_{\Omega_n} u_n > 0$ for every n and $(\theta_n)_{n\in\mathbb{N}}$ is bounded, then:*

$$\liminf_{n\to+\infty} \frac{1}{\theta_n} \int_{\Omega_n} f(\Phi_n(y), u_n(y), \nabla u_n(y))\, \mathrm{d}y \geq f'_-(x_0, 0, 0),$$

where $f'_-(x_0, 0, 0)$ was defined in (1.4).

Proof of Proposition 3.8. Without loss of generality, we may assume after extracting a subsequence that:

$$K := \sup_n \frac{1}{\theta_n} \int_{\Omega_n} f(\Phi_n(y), u_n(y), \nabla u_n(y))\, \mathrm{d}y + \theta_n < +\infty. \tag{3.18}$$

We consider the sequence of measures $(\nu_n)_{n\in\mathbb{N}} \in \mathcal{M}_+(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$ defined by

$$\nu_n := \frac{1}{\theta_n} (\Phi_n, u_n, \nabla u_n)_\sharp (u_n \mathcal{L}^N \llcorner \Omega_n), \quad n \in \mathbb{N}.$$

We are going to show in several steps that $\nu_n \xrightharpoonup{\mathcal{C}'_b} \delta_{(x_0,0,0)}$ and deduce the result. It suffices to show that the three projections $\nu_n^i := (\pi^i)_\sharp \nu_n$, $i \in \{1, 2, 3\}$ converge narrowly to δ_{x_0}, δ_0 and δ_0 respectively. Indeed, this would imply that (ν_n) converges narrowly to a measure concentrated on $(x_0, 0, 0)$, hence to $\delta_{(x_0,0,0)}$ since the ν_n are probability measures. First of all, since (ν_n) has bounded mass and (θ_n) is bounded, we may take a subsequence (not relabeled) such that $\nu_n \xrightharpoonup{\mathcal{C}'_0} \nu$ and $\theta_n \to \theta$ as $n \to \infty$ for some $\nu \in \mathcal{M}_+(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$ and $\theta \geq 0$.

Step 1: $\nu_n^1 \xrightharpoonup{\mathcal{C}'_b} \delta_{x_0}$. This is a direct consequence of the fact that ν_n^1 is concentrated on $\Phi_n(\mathbb{R}^N)$ for every n and $\mathrm{dist}(\Phi_n(\mathbb{R}^N), x_0)$ as $n \to \infty$.

Step 2: $\nu_n^2 \xrightharpoonup{\mathcal{C}'_b} \delta_0$. By (3.18) and our assumption (H4), there is a constant $K_1 > 0$ with

$$\int_{\Omega_n} |\nabla u_n|^p \leq K_1 \int_{\Omega_n} u_n, \quad \forall n \in \mathbb{N}. \tag{3.19}$$

We deduce from Markov's inequality, and Lemma 3.7 applied with $r = p(1 + \frac{1}{N})$, corresponding to $\alpha = \frac{N}{N+1}$, that

$$\begin{aligned}
\nu_n^2([\eta, +\infty)) = \frac{1}{\theta_n} \int_{\{u_n \geq \eta\}} u_n &= \frac{1}{\theta_n} \int_{\{u_n \geq \eta\}} u_n^{1-r} u_n^r \\
&\leq \frac{1}{\theta_n \eta^{r-1}} \int_{\Omega_n} u_n^r \\
&\leq \frac{C}{\theta_n \eta^{r-1}} (\|\nabla u_n\|_{L^p(\Omega_n)} + \|u_n\|_{L^1(\Omega_n)})^{r\alpha} \|u_n\|^{r(1-\alpha)}_{L^1_{\mathrm{uloc}}(\Omega_n)} \\
&\leq \frac{C'}{\eta^{r-1}} (1 + \theta_n^{p-1}) \|u_n\|^{r(1-\alpha)}_{L^1_{\mathrm{uloc}}(\Omega_n)},
\end{aligned}$$

where in the last inequality, we have used the identity $\alpha r = p$ and (3.19).

Since $(u_n)_{n\in\mathbb{N}}$ is vanishing and $(\theta_n)_{n\in\mathbb{N}}$ is bounded, the last term in the previous inequality goes to zero as $n \to \infty$ and it follows that $\nu_n^2 \xrightarrow{\mathcal{C}_b'} \delta_0$.

Step 3: $\nu_n^3 \xrightarrow{\mathcal{C}_b'} \delta_0$. Fix $M > 0$ and $\eta > 0$. One has by (3.19),

$$\begin{aligned}
\nu_n^3([M,+\infty)) = \frac{1}{\theta_n}\int_{\{|\nabla u_n|\geq M\}} u_n &\leq \frac{1}{\theta_n}\int_{\{u_n<\eta\}\cap\{|\nabla u_n|\geq M\}} u_n + \frac{1}{\theta_n}\int_{\{u_n>\eta\}} u_n \\
&\leq \frac{\eta}{\theta_n}\mathcal{L}^N(\{|\nabla u_n|\geq M\}) + \nu_n^2([\eta,+\infty)) \\
&\leq \frac{\eta}{\theta_n}\frac{1}{M^p}\int_{\Omega_n}|\nabla u_n|^p + \nu_n^2([\eta,+\infty)) \\
&\leq \frac{\eta K_1}{M^p} + \nu_n^2([\eta,+\infty)).
\end{aligned}$$

By the previous step, we know that $\lim_{n\to+\infty}\nu_n^2([\eta,+\infty)) = 0$, hence taking the superior limit as $n \to +\infty$ then $\eta \to 0$ we get $\lim_{n\to+\infty}\nu_n^3([M,+\infty)) = 0$. Since this is true for every $M > 0$ we obtain $\nu_n^3 \xrightarrow{\mathcal{C}_b'} \delta_0$.

Step 4: conclusion. By the previous steps, we deduce that $\nu_n \xrightarrow{\mathcal{C}_b'} \delta_{(x_0,0,0)}$ as $n \to +\infty$. We define $g : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to [0,+\infty]$ as the lower semicontinuous envelope of $\mathbb{R}^N \times \mathbb{R}_+^* \times \mathbb{R}^N \ni (x,u,\xi) \mapsto \frac{1}{u}f(x,u,\xi)$. By (H1), we have $g(x,u,\xi) = \frac{1}{u}f(x,u,\xi)$ if $u > 0$, and by (1.4), we have $g(x,0,0) = f'_-(x,0,0)$ for every $x \in \mathbb{R}^N$. Hence, by lower semicontinuty of g and Fatou's lemma, we get

$$\begin{aligned}
\liminf_{n\to\infty}\int_{\Omega_n} f(\Phi_n,u_n,\nabla u_n) &\geq \liminf_{n\to\infty}\int_{\{u_n>0\}}\frac{f(\Phi_n,u_n,\nabla u_n)}{u_n}u_n \\
&= \liminf_{n\to\infty}\int_{\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N} g(x,u,\xi)\,\mathrm{d}\nu_n(x,u,\xi) \\
&\geq \int_{\mathbb{R}^N} g(x,u,\xi)\,\mathrm{d}\delta_{(x_0,0,0)} = f'_-(x_0,0,0),
\end{aligned}$$

which ends the proof of the lemma. □

We now establish our main energy lower bound along sequences with bounded mass (not necessarily vanishing):

Proposition 3.9. *Assume that $(f_\varepsilon)_{\varepsilon>0}$ is a family of functions $f_\varepsilon : \mathbb{R}^N\times\mathbb{R}_+\times\mathbb{R}^N \to \mathbb{R}_+$ satisfying (H1), (H2), (H4) and (H6) for some limit f. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers going to zero, $(R_n)_{n\in\mathbb{N}}$ and $(r_n)_{n\in\mathbb{N}}$ be two sequences in $(0,+\infty]$ such that $\lim_{n\to\infty} r_n = \lim_{n\to\infty} R_n - r_n = +\infty$, $(u_n)_{n\in\mathbb{N}}$ be a sequence of functions $u_n \in W^{1,1}_{\mathrm{loc}}(B_{R_n},\mathbb{R}_+)$ with finite limit mass $m := \lim_{n\to\infty}\int_{B_{r_n}} u_n$, and $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence of Borel maps $\Phi_n : B_{R_n} \to \mathbb{R}^N$ such that*

$$\sup_{y\in B_{R_n}}|\Phi_n(y) - x_0| \xrightarrow[n\to\infty]{} 0 \quad \textit{for some } x_0 \in \mathbb{R}^N. \tag{3.20}$$

Then there exists a family $(u^i)_{0\le i<k}$ *of functions in* $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{R}_+)$ *with* $k\in\mathbb{N}\cup\{+\infty\}$, *such that* $m_i := \int_{\mathbb{R}^N} u^i \in (0,+\infty)$ *for every* i, *and*

$$m = m_v + \sum_{0\le i<k} m_i \quad \textit{with } m_v \ge 0, \tag{3.21}$$

$$\liminf_{n\to\infty} \int_{B_{R_n}} f_{\varepsilon_n}(\Phi_n, u_n, \nabla u_n) \ge m_v f'_-(x_0,0,0) + \sum_{0\le i<k} \int_{\mathbb{R}^N} f(x_0, u^i, \nabla u^i). \tag{3.22}$$

Proof. We first assume, up to subsequence, that the left hand side of (3.22) is a limit. We apply the profile decomposition Theorem 3.2 to the sequence of positive measures $\mu_n = u_n \mathcal{L}^N_{|B_{r_n}}$ where, without loss of generality, we assume the extraction σ to be the identity for convenience, and we use the same notation as in Theorem 3.2. In particular, for each bubble $B^i_n = B_{r^i_n}(x^i_n)$, with $0\le i<k_n$, we have $x^i_n \in \operatorname{supp}\mu_n \subseteq B_{r_n}$. By assumption, we have $\lim_{n\to\infty}(R_n - r_n) = +\infty$; hence, up to reducing the radii of the balls B^i_n if necessary, in such a way that their radii still diverge (see Remark 3.4), we can assume that

$$B^i_n \subseteq B_{R_n-1}, \quad 0\le i<k_n. \tag{3.23}$$

For each $0\le i<k_n$, we let $u^i_n := u_n(\cdot + x^i_n)$. Assuming without loss of generality that the left hand side of (3.22) is finite, we get that the sequence $(u^i_n)_n$ is bounded in $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$ by (H4). Hence, after a further extraction if needed, we get that $(u^i_n)_{n\in\mathbb{N}} \rightharpoonup u^i$ weakly in $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$ for some limit u^i, for every $0\le i<k=\lim k_n$. Setting $m_i = \int_{\mathbb{R}^N} u^i$ for every i, by (D) in Theorem 3.2, we have

$$m_v := m - \sum_{0\le i<k} m_i = \lim_{n\to\infty} \int_{B_{r_n}\setminus \cup_{0\le i<k_n} B^i_n} u_n.$$

Fix $\varepsilon>0$. We decompose the energy as

$$\int_{B_{R_n}} f_\varepsilon(\Phi_n,u_n,\nabla u_n) = \int_{B_{R_n}\setminus\cup_{0\le i<k_n}B^i_n} f_\varepsilon(\Phi_n,u_n,\nabla u_n) + \sum_{0\le i<k_n}\int_{B_{r^i_n}} f_\varepsilon(\Phi_n(\cdot+x^i_n),u^i_n,\nabla u^i_n). \tag{3.24}$$

Note that the domains $\Omega_n := B_{R_n}\setminus \cup_{0\le i<k} B^i_n$ satisfy $\inf_{n\in\mathbb{N}} \operatorname{reach}(\partial\Omega_n) > 0$ as noticed in Example 3.5, thanks to (3.23) and (B), (C) in Theorem 3.2. Hence, applying Proposition 3.8 to the Lagrangian f_ε, we obtain

$$\liminf_{n\to\infty} \int_{B_{R_n}\setminus\cup_{0\le i<k_n}B^i_n} f_\varepsilon(\Phi_n,u_n,\nabla u_n) \ge m_v (f_\varepsilon)'_-(x_0,0,0). \tag{3.25}$$

Moreover, by lower semicontinuity of integral functionals (see [But89, Theorem 4.1.1]), in view of (3.20), we have for each i with $0\le i<k$,

$$\liminf_{n\to\infty} \int_{B_{r^i_n}} f_\varepsilon(\Phi_n(\cdot+x^i_n),u^i_n,\nabla u^i_n) \ge \int_{\mathbb{R}^N} f_\varepsilon(x_0,u^i,\nabla u^i). \tag{3.26}$$

Finally, by (3.24), (3.25), (3.26), (H6) and by monotone convergence, we deduce that

$$\liminf_{n\to\infty}\int_{B_{R_n}} f_{\varepsilon_n}(\Phi_n,u_n,\nabla u_n) \geq \lim_{\varepsilon\to 0^+}\Big(m_v (f_\varepsilon)'_-(x_0,0,0)+\sum_{0\leq i<k}\int_{\mathbb{R}^N} f_\varepsilon(x_0,u^i,\nabla u^i)\Big)$$
$$= m_v f'_-(x_0,0,0)+\sum_{0\leq i<k}\int_{\mathbb{R}^N} f(x_0,u^i,\nabla u^i). \qquad \square$$

3.3 Existence of optimal profiles

For the existence of an optimal profile in (1.2), we need a criterion that rules out splitting and vanishing of minimizing sequences:

Lemma 3.10. *Let $H:\mathbb{R}_+\to\mathbb{R}_+$ be a concave function. Then H is subadditive, and if for some $0<\theta<m$ one has $H(m)=H(m-\theta)+H(\theta)$, then H is linear on $(0,m)$.*

Proof. By concavity, $t\mapsto \frac{H(t)}{t}$ is non-increasing. Hence,

$$H(m)=\theta\frac{H(m)}{m}+(m-\theta)\frac{H(m)}{m}\leq \theta\frac{H(\theta)}{\theta}+(m-\theta)\frac{H(m-\theta)}{m-\theta}.$$

But, by assumption, the last inequality is an equality which means that $\frac{H(m)}{m}=\frac{H(\theta)}{\theta}=\frac{H(m-\theta)}{m-\theta}$. In particular, the monotone function $t\mapsto\frac{H(t)}{t}$ must be constant on $[\theta,m]$, i.e. H must be linear on $[\theta,m]$. By concavity this is only possible if H is linear on $[0,m]$. $\square$

We can now state and prove our existence result:

Proposition 3.11. *Assume that $f:\mathbb{R}^N\times\mathbb{R}_+\times\mathbb{R}^N\to\mathbb{R}_+$ satisfies (H1), (H2), (H4) and (H5). Let $(x_0,m)\in\mathbb{R}^N\times\mathbb{R}_+$. If $H_f(x_0,\cdot)$ is not linear on $[0,m]$ then (1.2) admits a solution $u\in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{R}_+)$, i.e. $\int_{\mathbb{R}^N}u=m$ and $\int_{\mathbb{R}^N}f(x_0,u,\nabla u)=H_f(x_0,m)$.*

Proof. If $m=0$, we take $u=0$. If $m>0$, we apply Proposition 3.9 in the following situation: $f_\varepsilon\equiv f$, $R_n\equiv+\infty$, $\Phi_n\equiv x_0$, $(u_n)_{n\in\mathbb{N}}$ is a minimizing sequence for the minimization problem in (1.2), and $(r_n)_{n\in\mathbb{N}}$ is a sequence of positive radii going to $+\infty$ such that $\lim_{n\to\infty}\int_{B_{r_n}}u_n=m$. We obtain

$$H_f(x_0,m)\geq m_v f'_-(x_0,0,0)+\sum_{0\leq i<k}\int_{\mathbb{R}^N} f(x_0,u^i,\nabla u^i),$$

with $k\in\mathbb{N}\cup\{+\infty\}$, $u^i\in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{R}_+)$ and $m=\sum_{0\leq i<k}m_i+m_v$, where $m_i:=\int_{\mathbb{R}^N}u^i$. By Proposition 2.8 and Proposition 2.9, in view of our assumption (H5), we have $f'_-(x_0,0,0)\geq H'_f(x_0,0)$. Moreover, by lemma 2.4, we have $m_vH'_f(x_0,0)\geq H_f(x_0,m_v)$. Hence, by definition of H_f,

$$H_f(x_0,m)\geq m_vf'_-(x_0,0,0)+\sum_{0\leq i<k}\int_{\mathbb{R}^N}f(x_0,u^i,\nabla u^i)\geq\sum_{0\leq i<k}H_f(x_0,m_i)+H_f(x_0,m_v).$$

Since the concave function $H_f(x_0,\cdot)$ is not linear on $[0,m]$, by Lemma 3.10, we have either $k=1$ and $m_v=0$, and we are done, or $k=0$ and $m=m_v$. But in the latter case, we would have $H_f(x_0,m)=mH_f'(x_0,0)$ which implies that the monotone function $t\mapsto \frac{H_f(x_0,t)}{t}$ is constant on $[0,m]$, i.e. that $H_f(x_0,\cdot)$ is linear on $[0,m]$. This contradicts our assumption. □

4 Γ-convergence of the rescaled energies towards the H-mass

We establish lower and upper bounds for the $\Gamma-\liminf$ and $\Gamma-\limsup$ respectively, from which we deduce the proof of our main Γ-convergence result. The upper bound on the $\Gamma-\limsup$ holds under more general assumptions and will be needed in Section 5.5.

4.1 Lower bound for the $\Gamma-\liminf$

Given a Borel function $f:\mathbb{R}^N\times\mathbb{R}_+\times\mathbb{R}^N\to\mathbb{R}_+$, we define

$$H_f^-(x,m) := \inf\{H_f(x,m), f'_-(x,0,0)m\},\quad x\in\mathbb{R},\ m\in\mathbb{R}_+, \tag{4.1}$$

recalling that H_f is defined in (1.2) and $f'_-(x,0,0)$ in (1.4). Notice that under (H5), in view of Proposition 2.8 and Proposition 2.9 we have $H_f^-(x,m)=H_f(x,m)$.

Proposition 4.1. *Assume that $(f_\varepsilon)_{\varepsilon>0}$ is a family of functions $f_\varepsilon:\mathbb{R}^N\times\mathbb{R}_+\times\mathbb{R}^N\to\mathbb{R}_+$ satisfying (H1), (H2), (H4) and (H6) where $f=\lim_{\varepsilon\to0}f_\varepsilon$. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers going to zero, $(u_n)_{n\in\mathbb{N}}$ be a sequence in $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{R}_+)$, and let*

$$e_n(x) := f_{\varepsilon_n}(x,\varepsilon_n^N u_n(x),\varepsilon_n^{N+1}\nabla u_n(x))\varepsilon_n^{-N},\quad x\in\mathbb{R}^N,$$

be the energy density of u_n. If $u_n\mathcal{L}^N\xrightarrow{\mathcal{C}_0'} u\in\mathcal{M}_+(\mathbb{R}^N)$ and $e_n\mathcal{L}^N\xrightarrow{\mathcal{C}_0'} e\in\mathcal{M}_+(\mathbb{R}^N)$, then

$$e\geq H_-^f(u). \tag{4.2}$$

In particular, $\Gamma(\mathcal{C}_0')-\liminf_{\varepsilon\to0}\mathcal{E}_\varepsilon\geq \mathbf{M}^{H_f^-}$.

Proof of Proposition 4.1. Set $H:=H_f^-$. To obtain (4.2), it is enough to prove that if $x_0\in\mathbb{R}^N$ is an atom of u, i.e. $u(\{x_0\})>0$, then

$$e(\{x_0\})\geq H(x_0,u(\{x_0\})). \tag{4.3}$$

and that if $x_0\in\operatorname{supp}u$ is not an atom of u, then

$$\limsup_{R\to0^+}\frac{e(B_R(x_0))}{u(B_R(x_0))}\geq H'(x_0,0). \tag{4.4}$$

Indeed (4.3) implies that $e\geq(H(u))^a$ (the atomic part of the measure $H(u)$) while (4.4) implies that $e\geq H'(\cdot,0)u^d=(H(u))^d$, by Radon-Nikodỳm theorem (see [AFP00, Theorem 2.22]); these two relations yield $e\geq(H(u))^a+(H(u))^d=H(u)$ as required.

We fix $x_0 \in \operatorname{supp} u$ and proceed in several steps.

Step 1: blow-up near x_0. We first take two sequences of positive radii $(R_\ell)_{\ell\in\mathbb{N}} \to 0$ and $(r_\ell)_{\ell\in\mathbb{N}}$ such that for every $\ell \in \mathbb{N}$, $r_\ell \in (0, R_\ell)$,

$$e(\partial B_{R_\ell}(x_0)) = u(\partial B_{r_\ell}(x_0)) = 0, \tag{4.5}$$

and

$$\lim_{\ell\to\infty} \frac{e(B_{R_\ell}(x_0))}{u(B_{r_\ell}(x_0))} = \limsup_{R\to 0^+} \frac{e(B_R(x_0))}{u(B_R(x_0))}. \tag{4.6}$$

This last property is obtained by taking first a sequence $(\rho_\ell)_\ell$ such that

$$\limsup_{R\to 0^+} \frac{e(B_R(x_0))}{u(B_R(x_0))} = \lim_{\ell\to\infty} \frac{e(B_{\rho_\ell}(x_0))}{u(B_{\rho_\ell}(x_0))},$$

then using monotone convergence the measures to get first r_ℓ then R_ℓ such that $0 < r_\ell < R_\ell < \rho_\ell$, $u(B_{r_\ell}(x_0)) \geq (1-2^{-\ell})u(B_{\rho_\ell}(x_0))$ and $e(B_{R_\ell}(x_0)) \geq (1-2^{-\ell})e(B_{\rho_\ell}(x_0))$.

By weak convergence and (4.5), according to [AFP00, Proposition 1.62 b)], we have for every $\ell \in \mathbb{N}$,

$$\lim_{n\to\infty} e_n(B_{R_\ell}(x_0)) = e(B_{R_\ell}(x_0)) \quad \text{and} \quad \lim_{n\to\infty} \int_{B_{r_\ell}(x_0)} u_n = u(B_{r_\ell}(x_0)).$$

Hence, there exists an extraction $(n_\ell)_{\ell\in\mathbb{N}} \in \Sigma$ such that

$$\lim_{\ell\to\infty} \frac{r_\ell}{\varepsilon_{n_\ell}} = +\infty \quad \text{and} \quad \lim_{\ell\to\infty} \frac{R_\ell - r_\ell}{\varepsilon_{n_\ell}} = +\infty, \tag{4.7}$$

satisfying the following conditions:

$$u(\{x_0\}) = \lim_{\ell\to\infty} \int_{B_{r_\ell}(x_0)} u_{n_\ell}, \quad e(\{x_0\}) = \lim_{\ell\to\infty} e_{n_\ell}(B_{R_\ell}(x_0)), \tag{4.8}$$

and

$$\limsup_{\ell\to\infty} \frac{e(B_{R_\ell}(x_0))}{u(B_{r_\ell}(x_0))} = \lim_{\ell\to\infty} \frac{e_{n_\ell}(B_{R_\ell}(x_0))}{\int_{B_{r_\ell}(x_0)} u_{n_\ell}}. \tag{4.9}$$

We may rewrite the mass and energy in terms of the re-scaled map v_ℓ defined by

$$v_\ell(y) := \varepsilon_{n_\ell}^N u_{n_\ell}(x_0 + \varepsilon_{n_\ell} y), \quad y \in \mathbb{R}^N, \ell \in \mathbb{N} \tag{4.10}$$

as follows:

$$\int_{B_{r_\ell}(x_0)} u_{n_\ell} = \int_{B_{\varepsilon_{n_\ell}^{-1} r_\ell}} v_\ell, \tag{4.11}$$

and

$$e_{n_\ell}(B_{R_\ell}(x_0)) = \int_{B_{\varepsilon_{n_\ell}^{-1} R_\ell}} f_{\varepsilon_{n_\ell}}(x_0 + \varepsilon_{n_\ell} y, v_\ell(y), \nabla v_\ell(y))\, \mathrm{d}y. \tag{4.12}$$

Step 2: proof of (4.3). By Proposition 3.9, we have

$$e(\{x_0\}) = \lim_{\ell\to\infty} e_{n_\ell}(B_{R_\ell}(x_0)) \geq m_v f'_-(x_0,0,0) + \sum_{0\leq i<k} H_f(x_0,m_i). \tag{4.13}$$

Here $k \in \mathbb{N}\cup\{+\infty\}$ and $m = m_v + \sum_{0\leq i<k} m_i$, with $m_i > 0$, $m_v \geq 0$ and

$$m = \lim_{\ell\to\infty} \int_{B_{\varepsilon_{n_\ell}^{-1} r_\ell}} v_\ell = u(\{x_0\}).$$

Since the function $H = H_f^-$, defined in (4.1), is the infimum of two functions which are concave in the mass m, it is itself concave in m hence subadditive. From (4.13) we thus arrive at

$$e(\{x_0\}) \geq H_f^-(x_0,m_v) + \sum_{0\leq i<k} H_f^-(x_0,m_i) \geq H_f^-\Big(x_0, m_v + \sum_{0\leq i<k} m_i\Big) = H_f^-(x_0,u(\{x_0\})).$$

Step 3: proof of (4.4). Fix $\varepsilon > 0$ and assume that $m = u(\{x_0\}) = 0$. In that case, we apply Proposition 3.8 to the sequence of functions $(v_\ell)_{\ell\in\mathbb{N}}$ defined on the sets $\Omega_\ell = B_{\varepsilon_{n_\ell}^{-1} r_\ell}$ and the function f_ε to get, thanks to (H6):

$$\begin{aligned}
\limsup_{R\to 0^+} \frac{e(B_R(x_0))}{u(B_R(x_0))} &= \lim_{\ell\to\infty} \frac{e_{n_\ell}(B_{R_\ell}(x_0))}{\int_{B_{r_\ell}(x_0)} u_{n_\ell}} \\
&\geq \liminf_{\ell\to\infty} \frac{1}{\int_{B_{\varepsilon_{n_\ell}^{-1} r_\ell}} v_\ell} \int_{B_{\varepsilon_{n_\ell}^{-1} r_\ell}} f_\varepsilon(x_0 + \varepsilon_{n_\ell} y, v_\ell(y), \nabla v_\ell(y)) \\
&\geq (f_\varepsilon)'_-(x_0,0,0).
\end{aligned}$$

Taking the limit $\varepsilon \to 0^+$, we deduce by (H6) and (4.1):

$$\limsup_{R\to 0^+} \frac{e(B_R(x_0))}{u(B_R(x_0))} \geq f'_-(x_0,0,0) \geq (H_f^-)'(x_0,0). \tag{4.14}$$

In view of the discussion at the beginning of the proof, we have now proved (4.2).

Step 4: lower bound for the $\Gamma - \liminf$. We justify that (4.2) implies the lower bound $\Gamma(\mathcal{C}'_0) - \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon \geq \mathbf{M}^{H_f^-}$. Indeed, fix $u \in \mathcal{M}_+(\mathbb{R}^N)$ and consider a family $(u_\varepsilon)_{\varepsilon>0}$ weakly converging to u as $\varepsilon \to 0$. We need to show that $\mathbf{M}^H(u) \leq \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon)$. Assume without loss of generality that the inferior limit is finite and take a sequence of positive numbers $(\varepsilon_n)_{n\in\mathbb{N}} \to 0$ such that this inferior limit is equal to $\lim_{n\to\infty} \mathcal{E}_{\varepsilon_n}(u_{\varepsilon_n})$. Now the energy density e_n associated to $u_n = u_{\varepsilon_n}$ has bounded mass and up to extracting a subsequence one may assume that it converges weakly to some measure $e \in \mathcal{M}_+(\mathbb{R}^N)$. By the previous steps, $e \geq H(u)$, and by lower semicontinuity and monotonicity of the mass:

$$\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) = \liminf_{n\to\infty} \|e_n\| \geq \|e\| \geq \|H(u)\| = \mathbf{M}^H(u). \qquad \square$$

4.2 Upper bound for the $\Gamma - \limsup$

In this section, we introduce the following substitute for (H6), where $f, (f_\varepsilon)_{\varepsilon>0}$ are Borel maps from $\mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}^N$ to $\mathbb{R}_+$:

(U) there exists $C < +\infty$ such that for every $x, y \in \mathbb{R}^N$, $u \in \mathbb{R}_+$ and $\xi \in \mathbb{R}^N$,

$$\limsup_{\varepsilon\to 0^+} f_\varepsilon(x+\varepsilon y, u, \xi) \leq f(x,u,\xi) \quad \text{and} \quad f_\varepsilon(y,u,\xi) \leq Cf(x,u,\xi) \quad \forall \varepsilon > 0.$$

Proposition 4.2. *Assume that $f, (f_\varepsilon)_{\varepsilon>0}$ satisfy (U). If $u \in \mathcal{M}_+(\mathbb{R}^N)$, then there exists $(u_\varepsilon)_{\varepsilon>0} \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}_+)$ such that $u_\varepsilon \mathcal{L}^N \xrightarrow{\mathcal{C}_b'} u$ when $\varepsilon \to 0$ and which satisfies*

$$\limsup_{\varepsilon\to 0^+} \mathcal{E}_\varepsilon(u_\varepsilon) \leq \mathbf{M}^{H_{f,\mathrm{lsc}}}(u),$$

where $H_{f,\mathrm{lsc}} \leq H_f$ stands for the lower semicontinuous envelope of H_f, defined in (2.6). In other words, we have $\Gamma(\mathcal{C}_b') - \limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon \leq \mathbf{M}^{H_{f,\mathrm{lsc}}}$.

Proof of Proposition 4.2. Let $F = \Gamma(\mathcal{C}_b') - \limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon$. As an upper Γ-limit, F is sequentially lower semicontinuous in the narrow topology. Hence, by Proposition 2.7, it is enough to prove that $F(u) \leq \mathbf{M}^{H_f}(u)$ whenever u is finitely atomic. Let $u = \sum_{i=1}^k m_i \delta_{x_i}$ with $k \in \mathbb{N}$, $m_i \geq 0$, $x_i \in \mathbb{R}^N$, and assume without loss of generality that $x_i \neq x_j$ when $i \neq j$ and $\mathbf{M}^{H_f}(u) < +\infty$. Fix $\eta > 0$. For each $i = 1, \dots, k$, there exists $u_i \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}_+)$ such that $\int_{\mathbb{R}^N} u_i = m_i$ and $\int_{\mathbb{R}^N} f(x_i, u_i, \nabla u_i) \leq H(x_i, m_i) + \eta$. We define for every $i = 1, \dots, k$,

$$u^i_\varepsilon(x) = \varepsilon^{-N} u_i(\varepsilon^{-1}(x - x_i)), \quad x \in \mathbb{R}^N, \tag{4.15}$$

and

$$u_\varepsilon = \sup\{u^i_\varepsilon \ : \ i = 1, \dots, k\}, \tag{4.16}$$

which converge narrowly as measures to u as $\varepsilon \to 0$. We have by change of variables:

$$\mathcal{E}_\varepsilon(u_\varepsilon) \leq \sum_{i=1}^k \mathcal{E}_\varepsilon(u_\varepsilon, \{u^i_\varepsilon = u_\varepsilon\}) \leq \sum_{i=1}^k \mathcal{E}_\varepsilon(u^i_\varepsilon) = \sum_{i=1}^k \int_{\mathbb{R}^N} f_\varepsilon(x_i + \varepsilon x, u_i, \nabla u_i).$$

Using our assumption (U) and the dominated convergence theorem, one gets as $\varepsilon \to 0$:

$$F(u) \leq \limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) \leq \sum_{i=1}^k \int_{\mathbb{R}^N} f(x_i, u_i, \nabla u_i) \leq \sum_{i=1}^k H(x_i, m_i) + k\eta = \mathbf{M}^H(u) + k\eta.$$

The conclusion follows by arbitrariness of $\eta > 0$. $\square$

4.3 Proof of the main Γ-convergence result

We now explain how Theorem 1.2 follows from Proposition 4.1 and Proposition 4.2.

Proof of Theorem 1.2. The lower bound $\Gamma(\mathcal{C}_0') - \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon \geq \mathbf{M}^{H_f^-}$ follows from Proposition 4.1, and the upper bound $\Gamma(\mathcal{C}_b') - \limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon \leq \mathbf{M}^{H_{f,\mathrm{lsc}}}$ from Proposition 4.2, where the assumption (U) is a consequence of (H3) and (H6). By Proposition 2.8 and Proposition 2.9, thanks to our assumption (H5), we have $H_f^- = H_f$, and $H_f \geq H_{f,\mathrm{lsc}}$ by definition. The result follows because the weak topology is weaker than the narrow topology. $\square$

5 Examples, counterexamples and applications

5.1 Scale-invariant Lagrangians and necessity of assumption (H5)

Our assumption (H5) is not very standard, but we need a condition of this type in order to get Γ-convergence of the rescaled energies $\mathcal{E}_\varepsilon$ towards $\mathbf{M}_{H_f}$, as shown by the following class of scale-invariant Lagrangians:

$$f_\varepsilon(x,u,\xi) = f(u,\xi) \quad \text{with} \quad f(u,\xi) = \begin{cases} u^{p(\frac{1}{p^\star}-1)}|\xi|^p & \text{if } u > 0, \\ 0 & \text{else,} \end{cases} \tag{5.1}$$

where $p \in (1, N)$, $N \in \mathbb{N}^*$ and $p^\star := \frac{pN}{N-p}$. By straightforward computations, $\mathcal{E}_\varepsilon(u) = \mathcal{E}_f(u) := \int_{\mathbb{R}^N} f(u, \nabla u)$ for every $\varepsilon > 0$ and $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$ in that case.

Moreover, the associated cost function H_f is not trivial. Indeed, applying the Gagliardo–Nirenberg–Sobolev inequality,

$$\Big(\int_{\mathbb{R}^N} |v|^{p^\star}\Big)^{\frac{1}{p^\star}} \leq C\Big(\int_{\mathbb{R}^N} |\nabla v|^p\Big)^{\frac{1}{p}}, \quad \forall v \in L^{p^\star} \cap W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N),$$

to the function $v = u^{\frac{1}{p^\star}}$, we obtain that for every $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}_+)) \cap L^1(\mathbb{R}^N)$,

$$\Big(\int_{\mathbb{R}^N} u\Big)^{\frac{p}{p^\star}} \leq \Big(\frac{C}{p^\star}\Big)^p \int_{\{u>0\}} u^{\frac{p}{p^\star}-p}|\nabla u|^p = \Big(\frac{C}{p^\star}\Big)^p \mathcal{E}_f(u).$$

Hence, for every $m > 0$, we have $H_f(m) > 0$, and even $H_f(m) < +\infty$ since any function $u = v^{p^\star}$, with $v \in W^{1,p}(\mathbb{R}^N, \mathbb{R}_+)$, has finite energy. Replacing u by mu in the infimum defining H_f in (1.2), we actually obtain

$$H_f(m) = m^{1-\frac{p}{N}} H_f(1), \quad 0 < H_f(1) < +\infty. \tag{5.2}$$

In that case, it is clear that the Γ-limit of $\mathcal{E}_\varepsilon \equiv \mathcal{E}$ in the weak or narrow topology of $\mathcal{M}_+(\mathbb{R}^N)$, that is the lower semicontinuous relaxation of $\mathcal{E}_f$, does not coincide with $\mathbf{M}^{H_f}$; indeed, the first functional is finite on diffuse measures whose density has finite energy, while the second functional is always infinite for non-trivial diffuse measures since $H_f'(0) = +\infty$.

These scaling invariant Lagrangians are ruled out by our assumption (H5). All the other assumptions are satisfied except (H4). Note that the following perturbation of f,

$$\tilde{f}(u,\xi) = (1 + u^{p(\frac{1}{p^\star}-1)})|\xi|^p$$

satisfies all the assumptions except (H5), and provides a counterexample to the Γ-convergence. Indeed, $\mathbf{M}_{H_{\tilde{f}}} \geq \mathbf{M}_{H_f}$ is still infinite on diffuse measures, while (the relaxation of) $\mathcal{E}_{\tilde{f}}$ is finite for any diffuse measure whose density has finite energy.

We stress that an assumption like (H5) is actually needed, even for the lower semicontinuity of the function H_f – recall that if $\mathbf{M}_{H_f}$ is a Γ-limit, then it must be lower semicontinuous by [Bra02, Proposition 1.28], which in turn implies that the function H_f is lower semicontinuous by Proposition 2.7. Indeed, consider the Lagrangians

$$f(x,u,\xi) = (1 + u^{p(\frac{1}{p^\star}-1)})|\xi|^{p(x)},$$

with $p \in \mathcal{C}^0(\mathbb{R}^N, (1,N))$ such that $p(0) = p \in (1,N)$ and $p(x) > p$ when $x \neq 0$. Then, we have $H_f(0,m) = m^{1-\frac{p}{N}} H(1)$, but $H_f(x,\cdot) \equiv 0$ if $x \neq 0$ as can be easily seen via the change of function $\varepsilon^N u(\varepsilon\,\cdot)$, with $\varepsilon > 0$ small.

5.2 General concave costs in dimension one

It has been proved in [Wir19] that for any continuous concave function $H : \mathbb{R}_+ \to \mathbb{R}_+$ with $H(0) = 0$, there exists a function $c : \mathbb{R}_+ \to \mathbb{R}_+$ such that $c(0) = 0$, $u \mapsto \frac{c(u)}{u}$ is lower semicontinuous and non-increasing on $(0,+\infty)$, and for every $m \geq 0$,

$$H(m) = \inf\left\{ \int_{\mathbb{R}} |u'|^2 + c(u) \;:\; u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R},\mathbb{R}_+),\ \int_{\mathbb{R}} u = m \right\}.$$

The Lagrangians of the form $f_\varepsilon(x,u,\xi) = |\xi|^2 + c(u)$, in dimension $N = 1$, satisfy all our assumptions (H1)–(H6), hence our Γ-convergence result stated in Theorem 1.2 yields the Γ-convergence of the functionals

$$\mathcal{E}_\varepsilon(u) = \int_{\mathbb{R}} \varepsilon^3 |u'|^2 + \frac{c(\varepsilon u)}{\varepsilon}, \quad u \in W^{1,2}(\mathbb{R},\mathbb{R}_+),$$

towards $\mathbf{M}^H$ for both the weak and narrow convergence of measures. Therefore, we may find an elliptic approximation of any concave H-mass. Let us stress that c is determined in [Wir19] from H through several operations including a deconvolution problem, but no closed form solution is given in general; nonetheless, an explicit solution is provided if c is affine by parts.

However, in dimension $N \geq 2$, we have no positive or negative answer to the inverse problem, consisting in finding f satisfying our assumptions with $H_f = H$, for a given continuous concave function $H : \mathbb{R}_+ \to \mathbb{R}_+$ with $H(0) = 0$. Note that, unlike the one-dimensional case, we cannot reach a function H having a non-trivial plateau with a Lagrangian of the form $f(x,u,\xi) = |\nabla u|^p + c(u)$, with $p \in (1,+\infty)$ and $c : \mathbb{R}_+ \to \mathbb{R}_+$ lower semicontinuous, in dimension $N \geq 2$.

Indeed, assume by contradiction that $H_f(m) = h_0 \in (0, +\infty)$ for every $m \in [m_1, m_2]$, with $0 \le m_1 < m_2$. Then, we get that f satisfies all our assumptions ((H5) being satisfied with $\rho(u) = u^\alpha$ if $\alpha \in (\frac{1}{p}, 1 + \frac{1}{N})$, for example $\alpha = 1$), and we deduce by Proposition 3.11 that there exists $u \in W^{1,p}(\mathbb{R}^N, \mathbb{R}_+)$ such that $\mathcal{E}_f(u) = H_f(m_2)$ and $\int_{\mathbb{R}^N} u = m_2$. By the Pólya-–Szegö inequality, up to replacing u by its symmetric decreasing rearrangement, we can assume that $u(x) = u^\star(|x|)$ with $u^\star : \mathbb{R}_+ \to \mathbb{R}_+$ non-increasing. Removing a slice of the form $\{-\eta \le x_1 \le \eta\}$ to the function u, and gluing together the two portions on either side of this slice, we obtain a function with slightly less mass, if $\eta > 0$ is small, and with less energy; since H_f is constant on a left neighbourhood of m_2, this means that the energy of u on this slice must vanish and, in particular, that u is constant here. Since u is radial, this means that u is constant on $\mathbb{R}^N$, a contradiction with the fact that $\int_{\mathbb{R}^N} u = m_2 \in (0, +\infty)$.

5.3 Homogeneous costs in any dimension

In this section, we provide Lagrangians f to obtain the α-mass $\mathbf{M}^\alpha := \mathbf{M}^{t \mapsto t^\alpha}$ in any dimension N for a wide range of exponents, including super-critical exponents $\alpha \in \left(1 - \frac{1}{N}, 1\right]$. We consider for every $p \in [1, +\infty), s \in (-\infty, 1]$ and $N \in \mathbb{N}^*$, the energy defined for every $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}_+)$ by

$$\mathcal{E}_{N,ps}(u) := \int_{\mathbb{R}^N} f_{N,p,s}(u, \nabla u) := \int_{\mathbb{R}^N} |\nabla u|^p + u^s. \tag{5.3}$$

Notice that for $p > 1$, $f_{N,p,s}$ satisfies all our hypotheses (H1)–(H5) (without dependence on x), (H5) holding in dimension $N \ge 2$ with $\rho(t) = t$ for example. Thus by Theorem 1.2 the re-scaled energies Γ-converge to the $H_{f_{N,p,s}}$-mass, with $H_{f_{N,p,s}}$.

One may compute $H_{f_{N,p,s}}$ substituting u by v such that $u = m\lambda^N v(\lambda\cdot)$ in (1.2), where

$$\lambda = m^{\frac{s/p-1}{1+N-sN/p}}. \tag{5.4}$$

Straightforward computations give $\int_{\mathbb{R}^N} v = 1$ if $\int_{\mathbb{R}^N} u = m$, and

$$\mathcal{E}_{N,p,s}(u) = m^{\alpha(N,p,s)} \mathcal{E}_{N,p,s}(v), \qquad \text{where} \quad \alpha(N,p,s) = \frac{1 - \frac{s}{p} + \frac{s}{N}}{1 - \frac{s}{p} + \frac{1}{N}},$$

thus

$$H_{N,p,s}(m) = c_{N,p,s} m^{\alpha(N,p,s)}, \qquad \text{where} \quad c_{N,p,s} = H_{N,p,s}(1).$$

We look for cases when the cost is non-trivial, i.e. neither identically zero nor infinite on $(0, +\infty)$. Take an auxiliary exponent $q \in [1, +\infty)$ and $\alpha \in [0, 1]$ such that $1 = \alpha q + (1 - \alpha)s$. By Hölder inequality,

$$\int_{\mathbb{R}^N} u = \int_{\mathbb{R}^N} u^{\alpha q} u^{(1-\alpha)s} \le \left(\int_{\mathbb{R}^N} u^q\right)^\alpha \left(\int_{\mathbb{R}^N} u^s\right)^{1-\alpha}.$$

Moreover, choosing $q \in (1, p^\star)$ if $p < N$ and any $q \in (1, +\infty)$ if $p \geq N$, by the Gagliardo–Nirenberg–Sobolev inequality, for every $u \in W^{1,1}_{\mathrm{loc}} \cap L^1(\mathbb{R}^N, \mathbb{R}_+)$,

$$\left(\int_{\mathbb{R}^N} u^q\right)^{\frac{1}{q}} = \|u\|_{L^q} \leq C \|\nabla u\|_{L^p}^{\beta} \|u\|_{L^1}^{1-\beta},$$

with $\beta \in (0, 1)$ such that $\frac{1}{q} = \beta\left(\frac{1}{p} - \frac{1}{N}\right) + (1 - \beta)$. Hence,

$$\left(\int_{\mathbb{R}^N} u\right)^{1-q\alpha(1-\beta)} \leq C \, \left(\int_{\mathbb{R}^N} |\nabla u|^p\right)^{\frac{q\alpha\beta}{p}} \left(\int_{\mathbb{R}^N} u^s\right)^{1-\alpha},$$

and the cost is non-zero for every $m > 0$.

In the case $s \in [0, 1]$, any $u = v^r$ with $v \in \mathcal{C}^1_c(\mathbb{R}^N)$ is a competitor with finite energy, thus $\mathcal{E}_{N,p,s}$ is non-trivial for every $p \in [1, +\infty)$. In the case $s < 0$, consider the competitor $u : x \mapsto (1 - |x|)^\gamma_+$ for $\gamma > 0$ to be fixed later. Then $\int_{\mathbb{R}^N} |\nabla u|^p < +\infty$ if and only if $t \mapsto (1 - t)^{(\gamma-1)p}$ is integrable at 1^-, i.e. $(\gamma - 1)p > -1 \iff \gamma > 1 - 1/p$, and $\int_{\{u>0\}} u^s < +\infty$ if and only if $\gamma s > -1 \iff \gamma < -1/s$. Therefore, one may find $\gamma > 0$ satisfying both conditions, and ensure that $H_{f_{N,p,r,s}}$ is non-trivial, if

$$-p' < s < 0.$$

To summarize, we have shown that $H_{f_{N,p,s}}$ is non-trivial if:

$$s \in (-p', 1].$$

Since $\alpha = \alpha(N, p, s)$ is monotone in s, one may easily compute the range of α. If p and N are fixed, α ranges over $\left(\frac{N-1}{N+1+1/p}, 1\right]$ when $s \in (-p', 1]$. Notice that when $N = 1$ we obtain the whole range $\alpha \in (0, 1]$, and at least the range $\left[1 - \frac{2}{N+1}, 1\right]$ for every p in dimension $N \geq 2$. Finally, we obtain a range $\alpha \in \left(\frac{N-1}{N+2}, 1\right]$ when p ranges over $(1 + \infty]$ in dimension N.

5.4 Branched transport approximation: H-masses of normal 1-currents

Branched Transport is a variant of classical optimal transport (see [San15] and Section 4.4.2 therein for a brief presentation of branched transport, and [BCM09] for a vast exposition) where the transport energy concentrates on a network, i.e. a 1-dimensional subset of $\mathbb{R}^d$, which has a graph structure when optimized with prescribed source and target measures. It can be formulated as a minimal flow problem,

$$\min\left\{\mathbf{M}^H_1(w) \;:\; \operatorname{div}(w) = \mu^- - \mu^+\right\},$$

where $\mu^\pm$ are probability measures on $\mathbb{R}^d$, $H : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}_+$ is mass-subadditive, and the H-mass $\mathbf{M}^H_1$ is this time defined for finite vector measures $w \in \mathcal{M}(\mathbb{R}^d, \mathbb{R}^d)$ whose distributional divergence is also a finite measure; in the language of currents, it is called a

1-dimensional normal current. Any such measure may be decomposed into a 1-rectifiable part $\theta\xi \cdot \mathcal{H}^1 \llcorner \Sigma$ where $\theta(x) \geq 0$ and $\xi(x)$ is a unit tangent vector to Σ for $\mathcal{H}^1$-a.e. $x \in \Sigma$, and a 1-diffuse part $w^\perp$ satisfying $|w^\perp|(A) = 0$ for every 1-rectifiable set A:

$$w = \theta\xi \cdot \mathcal{H}^1_{|M} + w^\perp.$$

The H-mass is then defined by:

$$\mathbf{M}_1^H(w) \coloneqq \int_\Sigma H(x, \theta(x))\,\mathrm{d}\mathcal{H}^1(x) + \int_{\mathbb{R}^d} H'(x, 0)\,\mathrm{d}|w^\perp|. \tag{5.5}$$

In the case $H(x,m) = m^\alpha$ with $0 < \alpha < 1$, a family of approximations of these functional has been introduced in [OS11]:

$$\mathcal{E}_{\beta,\varepsilon}(w) = \begin{cases} \int_{\mathbb{R}^d} \varepsilon^{\gamma_1}|\nabla w|^2 + \varepsilon^{-\gamma_2}|w|^\beta & \text{if } w \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^d), \\ +\infty & \text{otherwise,} \end{cases} \tag{5.6}$$

with $\beta = \frac{2-2d+2\alpha d}{3-d+\alpha(d-1)}$, $\gamma_1 = (d-1)(1-\alpha)$ and $\gamma_2 = 3-d+\alpha(d-1)$. It has been shown in [OS11; Mon17] that the functionals $\mathcal{F}_{\beta,\varepsilon}$ Γ-converge as $\varepsilon \to 0^+$, in the topology of weak convergence of u and its divergence, to a non-trivial multiple of the α-mass $\mathbf{M}_1^\alpha \coloneqq \mathbf{M}_1^H$ with $H(x,m) = m^\alpha$ in dimension $d = 2$. The result extends to any dimension d, by [Mon15], thanks to a slicing method that relates the energy $\mathcal{E}_{\beta,\varepsilon}$ with the energy of the sliced measures $u = (w \cdot \nu)_+$ supported on the slices $V_a = \{x \in \mathbb{R}^d \ : \ x \cdot \nu = a\} \simeq \mathbb{R}^N$, for any given unit vector $\nu \in \mathbb{R}^d$, defined by

$$\bar{\mathcal{E}}_{\beta,\varepsilon}(u) = \int_{\mathbb{R}^N} \varepsilon^{\gamma_1}|\nabla u|^2 + \varepsilon^{-\gamma_2}|u|^\beta.$$

The functionals $\bar{\mathcal{E}}_{\beta,\varepsilon}$ Γ-converge as $\varepsilon \to 0^+$, in the weak-$\star$ topology of $\mathcal{C}_b'$, to $c\mathbf{M}^\alpha$ for some non-trivial c, as shown in Section 5.3, and one may recover every α-mass in this way for $\alpha \in \left(\frac{2d-4}{2d+1}, 1\right]$, and in particular every so-called super-critical exponents for Branched Transport in dimension d, that is $\alpha \in (1 - 1/d, 1]$.

The same slicing method would allow to extend our Γ-convergence result stated in Theorem 1.2 to functionals defined on vector measure

$$\mathcal{E}_\varepsilon(w) = \begin{cases} \int_{\mathbb{R}^d} f_\varepsilon(x, \varepsilon^{d-1}|w|(x), \varepsilon^d|\nabla w|(x))\varepsilon^{1-d}\,\mathrm{d}x & \text{if } w \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^d), \\ +\infty & \text{otherwise,} \end{cases} \tag{5.7}$$

for Lagrangians $f_\varepsilon \to f$ fitting the framework of Theorem 1.2. The expected Γ-limit, for the weak topology of measures and their divergence measure, would be the functional $\mathbf{M}_1^{H_f}$, with H_f defined in (1.2). Note that this approach would provide approximations of H-masses for more general continuous and concave cost functions $H : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $H(0) = 0$. By [Wir19], we would obtain all such H-masses when $N = 1$ (corresponding to $d = 2$).

5.5 A Cahn-Hilliard model for droplets

Following the works [BDS96] in the one-dimensional case and [Dub98] in higher dimension, we consider functionals on $\mathcal{M}_+(\mathbb{R}^N)$ of the form:

$$\mathcal{W}_\varepsilon(u) = \begin{cases} \int_{\mathbb{R}^N} \varepsilon^{-\rho}(W(u) + \varepsilon|\nabla u|^2) & \text{if } u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}_+), \\ +\infty & \text{otherwise,} \end{cases} \tag{5.8}$$

where $W : \mathbb{R}_+ \to \mathbb{R}_+$ is a Borel function satisfying $W(t) \sim_{u\to+\infty} u^s$ for some exponent $s \in (-\infty, 1)$. In [BDS96; Dub98], it is in particular proven, under some assumptions on the slope of W at 0 and its regularity, that the family $(\mathcal{W}_\varepsilon)_{\varepsilon>0}$ Γ-converges to a non-trivial multiple of the α-mass, $\alpha = \frac{1-s/2+s/N}{1-s/2+1/N}$, when $s \in (-2, 1)$ and $\rho = \rho(s, N) := \frac{N(1-s)}{(N+2)+N(1-s)}$. In this section, we recover this Γ-convergence result using our general model.

Replacing ε with $\bar\varepsilon := \varepsilon^{(N+2)+N(1-s)}$ and noticing that $1-\rho = \frac{N+2}{(N+2)+N(1-s)}$, one gets for every $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}_+)$:

$$\begin{aligned}\mathcal{W}_{\bar\varepsilon}(u) = \int_{\mathbb{R}^N} \varepsilon^{-N(1-s)} W(u) + \varepsilon^{N+2}|\nabla u|^2 &= \int_{\mathbb{R}^N} \left([\varepsilon^{Ns} W(\varepsilon^{-N}\varepsilon^N u)] + |\varepsilon^{N+1}\nabla u|^2\right)\varepsilon^{-N} \\ &= \int_{\mathbb{R}^N} f^W_\varepsilon(x, \varepsilon^N u, \varepsilon^{N+1}\nabla u)\varepsilon^{-N},\end{aligned}$$

where f^W_ε is defined for every $x \in \mathbb{R}^N, u \in \mathbb{R}_+, \xi \in \mathbb{R}^N$ by

$$f^W_\varepsilon(x, u, \xi) := W_\varepsilon(u) + |\xi|^2 \quad \text{and} \quad W_\varepsilon(u) := \varepsilon^{Ns} W(\varepsilon^{-N} u).$$

Therefore if we take $f_\varepsilon = f^W_\varepsilon$ in our general model (1.3) we exactly get $\mathcal{W}_{\bar\varepsilon} = \mathcal{E}_\varepsilon$. The fact that $W(u) \sim u^s$ as $u \to +\infty$ implies that W_ε converges pointwise to the map $k_s : u \mapsto u^s$ if $u > 0$, $k_s(0) = 0$, hence f^W_ε converges to $f_s : (x, u, \xi) \mapsto k_s(u) + |\xi|^2$.

Theorem 5.1. *Assume that $W : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies:*

(HW1) *W is lower semicontinuous,*

(HW2) $\{W = 0\} = \{0\}$,

(HW3) $W(u) \sim_{u\to+\infty} u^s$ *for some* $s \in (-\infty, 1)$,

(HW4) $\sup_{u>0} \dfrac{W(u)}{u^s} < +\infty$,

(HW5) $0 < \liminf_{u\to0^+} \dfrac{W(u)}{u}$.

Then $(\mathcal{W}_\varepsilon)_{\varepsilon>0}$ Γ-converges to $\mathbf{M}^{H_{f_s}}$, for both topologies $\mathcal{C}'_0$ and $\mathcal{C}'_b$, and if $s \in (-2, 1]$ then $\mathbf{M}^{H_{f_s}}$ is a nontrivial multiple of $\mathbf{M}^\alpha$ where $\alpha = \frac{1-s/2+s/N}{1-s/2+1/N}$.

To prove this theorem, we start with a simple lemma.

Lemma 5.2. *Assume that W satisfies (HW1)–(HW5). Then for every $\delta \in (0,1)$, there exists $c_\delta \in (0,+\infty)$ such that for every $\varepsilon > 0$ and every $u \in \mathbb{R}_+$,*

$$\delta(u^p \wedge c_\delta \varepsilon^{-N(1-s)} u) \le W_\varepsilon(u). \tag{5.9}$$

Proof. Fix $\delta \in (0,1)$. There exists $M > 0$ such that $\delta u^s \le W(u)$ for every $u \ge M$. Besides, the map $w : u \mapsto W(u)/u$ is lower semicontinuous and positive on $(0, M]$ by (HW1) and (HW2), and since $\liminf_{u\to 0} w(u) > 0$ by (HW5), w is necessarily bounded from below on $(0, M]$ by some contant $c > 0$. As a consequence $W_\varepsilon(u) \ge \delta u^s$ if $u \ge \varepsilon^N M$ and $W_\varepsilon(u) \ge c\varepsilon^{N(s-1)} u$ if $u \le \varepsilon^N M$, hence:

$$\forall u \in \mathbb{R}, \quad W_\varepsilon(u) \ge \delta(u^s \wedge c\varepsilon^{-N(1-s)} u). \qquad \square$$

Proof of Theorem 5.1. By (HW4), there exists a constant C such that $f_\varepsilon^W \le C f_s$ for every ε, and since f_ε^W does not depend on the x variable and converges pointwise to f_s, (U) is satisfied and our $\Gamma - \limsup$ result stated in Proposition 4.2 yields

$$\mathbf{M}^{H_{f_s}} \ge \Gamma(\mathcal{C}'_b) - \limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon.$$

Fix $\delta \in (0,1)$. By Lemma 5.2, there exists c_δ such that

$$\forall x, u, \xi, \quad f_\varepsilon^W(x,u,\xi) \ge \delta(|\xi|^2 + (u^s \wedge c_\delta \varepsilon^{-N(1-s)} u) =: f_\varepsilon^\delta(x,u,\xi).$$

It is easy to check that f_ε^δ satisfies (H1), (H2) and (H4) for every $\varepsilon > 0$. Moreover $f_\varepsilon^\delta \uparrow \delta f_s$ and $(f_\varepsilon^\delta)'_-(\cdot,0,0) = \delta c_\delta \varepsilon^{-N(1-s)} \uparrow +\infty = (\delta f_s)'_-(\cdot,0,0)$ as $\varepsilon \to 0$, thus (H6) holds for the family $(f_\varepsilon^\delta)_{\varepsilon>0}$, and by applying our $\Gamma - \liminf$ result stated in Proposition 4.1 to the energies $\mathcal{E}_\varepsilon^\delta$ induced by f_ε^δ we get:

$$\Gamma(\mathcal{C}'_0) - \liminf \mathcal{E}_\varepsilon \ge \Gamma(\mathcal{C}'_0) - \liminf \mathcal{E}_\varepsilon^\delta \ge \mathbf{M}^{H^-_{\delta f_s}}.$$

We get the result by taking the limit $\delta \to 1$, noticing that $(f_s)'_-(\cdot,0,0) = +\infty$, so that $H^-_{\delta f_s} = H_{\delta f_s} = \delta H_{f_s}$ and $\mathbf{M}^{H^-_{\delta f_s}} = \mathbf{M}^{\delta H_{f_s}} = \delta \mathbf{M}^{H_{f_s}}$. $\square$

Remark 5.3. We recover the Γ-convergence results of [BDS96] and [Dub98] when $s \in (-2,1)$ under slightly more general assumptions: besides (HW2) and (HW3), the authors impose the existence of a nontrivial slope $\lim_{u\to 0} \frac{W(u)}{u} \in (0,+\infty)$ and a regularity condition (either W is of class $\mathcal{C}^1$ or continuous and nondecreasing close to 0), which are stronger than (HW1), (HW4) and (HW5). Let us stress however that these works also tackle the cases $s < -2$ in any dimension, where the exponent ρ has to be fixed to $\rho(-2,N)$, and the case $s = -2$ in dimension one, where a logarithmic factor must be introduced, replacing $\varepsilon^{-\rho}$ with $\varepsilon^{-\rho(-2,1)}|\log\varepsilon|^{-1} = \varepsilon^{-1/2}|\log\varepsilon|^{-1}$. This implies that in our model we get a trivial Γ-limit when $s \le -2$, namely $H_{f_s} \equiv +\infty$ on $(0,+\infty)$.

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