A necessary condition in a De Giorgi type conjecture for elliptic systems in infinite strips

Radu Ignat [*] Antonin Monteil [†]

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Dedicated to Haïm Brezis on his seventy-fifth anniversary with esteem

Abstract

Given a bounded Lipschitz domain $\omega \subset \mathbb{R}^{d-1}$ and a lower semicontinuous function $W : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ that vanishes on a finite set and that is bounded from below by a positive constant at infinity, we show that every map $u : \mathbb{R} \times \omega \to \mathbb{R}^N$ with

$$\int_{\mathbb{R}\times\omega} \left(|\nabla u|^2 + W(u)\right) \mathrm{d}x_1 \, \mathrm{d}x' < +\infty$$

has a limit $u^\pm \in \{W = 0\}$ as $x_1 \to \pm\infty$. The convergence holds in $L^2(\omega)$ and almost everywhere in ω.

Keywords. Nonlinear elliptic systems of PDEs; De Giorgi conjecture; Energy estimates.

1 Introduction

Let $N \geq 1$, $d \geq 2$ and $\Omega = \mathbb{R} \times \omega$ be an infinite cylinder in $\mathbb{R}^d$, where $\omega \subset \mathbb{R}^{d-1}$ is an open connected bounded set with Lipschitz boundary. For a lower semicontinuous potential $W : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$, we consider the functional

$$E(u) = \int_\Omega \Big(|\nabla u|^2 + W(u)\Big) \, \mathrm{d}x, \quad u \in \dot{H}^1(\Omega, \mathbb{R}^N), \tag{1.1}$$

where $|\cdot|$ is the euclidean norm and

$$\dot{H}^1(\Omega, \mathbb{R}^N) = \left\{ u \in H^1_{loc}(\Omega, \mathbb{R}^N) \ : \ \nabla u = (\partial_j u_i)_{1\leq i\leq N, 1\leq j\leq d} \in L^2(\Omega, \mathbb{R}^{N\times d}) \right\}.$$

A natural problem consists in studying optimal transition layers for the functional E between two wells $u^\pm$ of W (i.e., $W(u^\pm) = 0$). In particular, motivated by the De Giorgi conjecture, one aim is to analyse under which conditions on the potential W and on the dimensions d and N, every minimizer u of E connecting $u^\pm$ as $x_1 \to \pm\infty$ is one-dimensional, i.e., depending only on x_1.

*Institut de Mathématiques de Toulouse & Institut Universitaire de France, UMR 5219, Université de Toulouse, CNRS, UPS IMT, F-31062 Toulouse Cedex 9, France. Email: Radu.Ignat@math.univ-toulouse.fr

†Institut de Recherche en Mathématique et Physique, Université catholique de Louvain, École de Mathématique, Chemin du Cyclotron 2, bte L7.01.02, 1348 Louvain-la-Neuve, Belgium. Email: Antonin.Monteil@uclouvain.be

Obviously, such one-dimensional transition layers u coincide with their x'-average $\overline{u} : \mathbb{R} \to \mathbb{R}^N$ defined as

$$\overline{u}(x_1) := \fint_\omega u(x_1, x')\, \mathrm{d}x', \quad x_1 \in \mathbb{R}, \tag{1.2}$$

where $x' = (x_2, \ldots, x_d)$ denotes the $d-1$ last variables in ω and the x'-average symbol is denoted by $\fint_\omega = \frac{1}{|\omega|} \int_\omega$.

1.1 Main result

The purpose of this note is to prove a necessary condition for finite energy configurations u provided that W satisfies the following two conditions:

(H1) W has a finite number of wells, i.e., $\left|\{z \in \mathbb{R}^N : W(z) = 0\}\right| < \infty$;

(H2) $W_\infty := \liminf_{|z|\to\infty} W(z) > 0$.

More precisely, we prove that under these assumptions, there exist two wells $u^\pm$ of W such that $u(x_1, \cdot)$ converges to $u^\pm$ in L^2 and a.e. in ω as $x_1 \to \pm\infty$; in particular, the x'-average $\overline{u}$ (as a continuous map in $\mathbb{R}$) admits the limits $\overline{u}(\pm\infty) = u^\pm$ as $x_1 \to \pm\infty$. Here, $u(x_1, \cdot)$ stands for the trace of the Sobolev map $u \in \dot{H}^1(\Omega, \mathbb{R}^N)$ on the section $\{x_1\} \times \omega$ for every $x_1 \in \mathbb{R}$.

Theorem 1. *Let $\Omega = \mathbb{R} \times \omega$, where $\omega \subset \mathbb{R}^{d-1}$ is an open connected bounded set with Lipschitz boundary. If $W : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ is a lower semicontinuous potential satisfying* **(H1)** *and* **(H2)***, then every $u \in \dot{H}^1(\Omega, \mathbb{R}^N)$ with $E(u) < \infty$ connects two wells*[1] *$u^\pm \in \mathbb{R}^N$ of W at $x_1 = \pm\infty$ (i.e., $W(u^\pm) = 0$) in the sense that*

$$\lim_{x_1\to\pm\infty} \|u(x_1, \cdot) - u^\pm\|_{L^2(\omega,\mathbb{R}^N)} = 0 \quad \text{and} \quad \lim_{x_1\to\pm\infty} u(x_1, \cdot) = u^\pm \quad \text{for a.e. } x_1 \in \omega. \tag{1.3}$$

In particular,

$$\lim_{x_1\to\pm\infty} \fint_\omega u(x_1, x')\ \mathrm{d}x' = u^\pm.$$

Remark 2. i) As a consequence of the Poincaré-Wirtinger inequality[2], there exist two sequences $(R_n^+)_{n\in\mathbb{N}}$ and $(R_n^-)_{n\in\mathbb{N}}$ such that $(R_n^\pm)_{n\in\mathbb{N}} \to \pm\infty$ and

$$\|u(R_n^\pm, \cdot) - u^\pm\|_{H^1(\omega,\mathbb{R}^N)} \underset{n\to\infty}{\longrightarrow} 0 \tag{1.4}$$

(see [24, Lemma 3.2]).

ii) Theorem 1 also holds true if ω is a closed (i.e., compact, connected without boundary) Riemannian manifold.

iii) We have proved a version of Theorem 1 for divergence-free maps u (in the case $d = N$) in our paper [24] (see [24, Lemma 3.7]). More precisely, in that context, we proved that the x'-average map $\bar{u}$ admits limits $u^\pm$ as $x_1 \to \pm\infty$, where $u^\pm$ are two wells of W having the same component in the direction x_1.

[1] u^- and u^+ could be equal.

[2] The assumption that ω is connected with Lipschitz boundary is needed for the Poincaré-Wirtinger inequality.

1.2 Motivation

Our main result is motivated by the well-known De Giorgi conjecture that consists in investigating the one-dimensional symmetry of critical points of the functional E, i.e., solutions $u : \Omega \to \mathbb{R}^N$ to the nonlinear elliptic system

$$\begin{cases} \Delta u = \frac{1}{2}\nabla W(u) & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega = \mathbb{R} \times \partial\omega, \end{cases} \tag{1.5}$$

where W is assumed to be locally Lipschitz in (1.5) and ν is the unit outer normal vector field at $\partial\omega$. Theorem 1 states in particular that solutions u of finite energy satisfy the boundary condition (1.3) for some two wells $u^\pm$ of W. A natural question related to the De Giorgi conjecture arises in this context:

Question: Under which assumptions on the potential W and the dimensions d and N, is it true that every global minimizer u of E connecting two wells[3] of W is one-dimensional symmetric, i.e., $u = u(x_1)$?

Link with the Gibbons and De Giorgi conjectures. i) In the scalar case $N = 1$ (d is arbitrary) and $W(u) = \frac{1}{2}(1-u^2)^2$, the answer to the above question is positive provided that the limits (1.3) is replaced by uniform convergence (see [12, 17]); within these uniform boundary conditions, the problem is called the Gibbons conjecture. We mention that many articles have been written on Gibbons' conjecture in the case of the entire space $\Omega = \mathbb{R}^d$: more precisely, if a solution[4] $u : \mathbb{R}^d \to \mathbb{R}$ of the PDE

$$\Delta u = \frac{1}{2}\frac{dW}{du}(u) \quad \text{in} \quad \mathbb{R}^d \tag{1.6}$$

satisfies the convergence $\lim_{x_1 \to \pm\infty} u(x_1, x') = \pm 1$ uniformly in x' and $|u| \leq 1$ in $\mathbb{R}^d$, then u is one-dimensional (see [5, 6, 11, 18]). Let us now speak about the long standing De Giorgi conjecture. It predicts that any bounded solution u of (1.6) that is monotone in the x_1 variable is one-dimensional in dimension $d \leq 8$, i.e., the level sets $\{u = \lambda\}$ of u are hyperplanes. The conjecture has been solved in dimension $d = 2$ by Ghoussoub-Gui [21], using a Liouville-type theorem and monotonicity formulas. Using similar techniques, Ambrosio-Cabré [4] extended these results to dimension $d = 3$, while Ghoussoub-Gui [22] showed that the conjecture is true for $d = 4$ and $d = 5$ under some antisymmetry condition on u. The conjecture was finally proved by Savin [31] in dimension $d \leq 8$ under the additional condition $\lim_{x_1 \to \pm\infty} u(x_1, x') = \pm 1$ pointwise in $x' \in \mathbb{R}^{d-1}$, the proof being based on fine regularity results on the level sets of u. Lately, Del Pino-Kowalczyk-Wei [13] gave a counterexample to the De Giorgi conjecture in dimension $d \geq 9$, which satisfies the pointwise limit conditions $\lim_{x_1 \to \pm\infty} u(x_1, x') = \pm 1$ for a.e. $x' \in \mathbb{R}^{d-1}$. It would be interesting to investigate whether these results transfer (or not) to the context of the strip $\Omega = \mathbb{R} \times \omega$. We recall that the pointwise convergence is a necessary condition in the context of a strip $\mathbb{R} \times \omega$ (see Theorem 1).

ii) Less results are available for the vector-valued case $N \geq 2$. In the case $\Omega = \mathbb{R}^d$, $N = 2$ and $W(u_1, u_2) = \frac{1}{2}(u_1^2 - 1)^2 + \frac{1}{2}(u_2^2 - 1)^2 + \Lambda u_1^2 u_2^2 - \frac{1}{2}$ with $\Lambda \geq 1$ (so $W \geq 0$ and W has exactly four wells $\{(0, \pm 1), (\pm 1, 0)\}$, thus, **(H1)** and **(H2)** are satisfied), the Gibbons and De Giorgi conjectures corresponding to the system (1.5) are discussed in [19]. Several other phase separation models (e.g., arising in a binary mixture of Bose-Einstein condensates) are studied in the vectorial case where W has a non-discrete set of zeros (see e.g., [7, 8, 20]).

We recall that in the study of the De Giorgi conjecture for (1.6), i.e., $N = 1$, there is a link between monotonicity of solutions (e.g., the condition $\partial_1 u > 0$), stability (i.e., the second

[3] We say that u connects two wells $u^\pm$ of W if (1.3) is satisfied.

[4] Here, u needs not be a global minimizer of E within the boundary condition (1.3), nor monotone in x_1, i.e., $\partial_1 u > 0$. Obviously, this result applies also to global minimizers, as $|u| \leq 1$ in $\mathbb{R}^d$ by the maximum principle.

variation of the corresponding energy is nonnegative), and local minimization of the energy (in the sense that the energy does not decrease under compactly supported perturbations of u). We refer to [2, Section 4] for a fine study of these properties. In particular, it is shown that the monotonicity condition in the De Giorgi conjecture implies that u is a local minimizer of the energy (see [2, Theorem 4.4]). Therefore, it is natural to study the above question under the monotonicity condition in x_1 (instead of the global minimality condition on u).

Link with micromagnetic models. We have studied the above Question in the context of divergence-free maps $u : \mathbb{R}\times\omega \to \mathbb{R}^N$ where $d = N$ and $\omega = \mathbb{T}^{d-1}$ is the $(d-1)$-dimensional torus, see [24]. By developing a theory of calibrations, we have succeeded to give sufficient conditions on the potential W in order that the answer to Question is positive, in particular in the case where **(H1)** and **(H2)** are satisfied, see [24, Theorem 2.11]. In that context, Question is related to some reduced model in micromagnetics in the regime where the so-called stray-field energy is strongly penalized favoring the divergence constraint $\nabla\cdot u = 0$ of the magnetization u (the unit-length constraint on u being relaxed in the system). In the theory of micromagnetics, a challenging question concerns the symmetry of domain walls. Indeed, much effort has been devoted lately to identifying on the one hand, the domain walls that have one-dimensional symmetry, such as the so-called symmetric Néel and symmetric Bloch walls (see e.g. [14, 26, 23]), and on the other hand, the domain walls involving microstructures, such as the so-called cross-tie walls (see e.g., [3, 30]), the zigzag walls (see e.g., [25, 29]) or the asymmetric Néel / Bloch walls (see e.g. [16, 15]). Thus, answering to Question would give a general approach in identifying the anisotropy potentials W for which the domain walls are one-dimensional in the elliptic system (1.5).

Link with heteroclinic connections. One dimensional solutions $u = u(x_1)$ of the system [5] (1.5) are called heteroclinic connections. Given two wells $u^\pm$ of a potential W satisfying **(H1)** and **(H2)**, it is known that there exists a heteroclinic connection $\gamma : \mathbb{R} \to \mathbb{R}^N$ obtained by minimizing $\int_{\mathbb{R}} |\frac{d}{dx_1}\gamma|^2 + W(\gamma)\,dx_1$ under the condition $\gamma(\pm\infty) = u^\pm$ (see [27, 33, 34]). In the vectorial case $N \geq 2$, this connection may not be unique in the sense that there could exist two (minimizing) heteroclinic connections γ_1, γ_2 such that $\gamma_i(\pm\infty) = u^\pm$ for $i = 1, 2$ but $\gamma_1(\cdot)$ and $\gamma_2(\cdot - \tau)$ are distinct for every $\tau \in \mathbb{R}$. If this is the case, at least in dimension $d = 2$ and $\Omega = \mathbb{R}^2$, there also exists a solution u to $\Delta u = \frac{1}{2}\nabla W(u)$ which realizes an interpolation between γ_1 and γ_2 in the following sense (see [32, 1, 28]):

$$\begin{cases} u(x_1, x_2) \to u^\pm & \text{as } x_1 \to \pm\infty \text{ locally uniformly in } x_2,\\ u(x_1, x_2) \to \gamma_1(x_1) & \text{as } x_2 \to -\infty \text{ locally uniformly in } x_1,\\ u(x_1, x_2) \to \gamma_2(x_1) & \text{as } x_2 \to +\infty \text{ locally uniformly in } x_1.\end{cases}$$

Moreover, this solution is a local minimizer of the energy. Solutions to the system $\Delta u = \frac{1}{2}\nabla W(u)$ naturally arise when looking at the local behavior of a transition layer near a point at the interface between two wells $u^\pm$; solutions satisfying the preceding boundary conditions correspond to the case of an interface point where the 1D connection passes from γ_1 to γ_2. The existence of such stable entire solutions to the Allen-Cahn system makes a significative difference with the scalar case, i.e. $N = 1$, where only 1D solutions are present by the De Giorgi conjecture.

2 Pointwise convergence and convergence of the x'-average

In this section we prove that under the assumptions in Theorem 1, the x'-average $\overline{u}$ (as a continuous map in $\mathbb{R}$) has limits $\overline{u}(\pm\infty) = u^\pm$ as $x_1 \to \pm\infty$ corresponding to two wells of W. For that, we

[5] If $u = u(x_1)$, the Neumann condition $\frac{\partial u}{\partial \nu} = 0$ is automatically satisfied.

will follow the strategy that we developed in our previous paper (see [24, Section 3.1]). The idea consists in introducing a new functional E_V associated to the x'-average $\overline{u}$ of a map u such that $\frac{1}{|\omega|}E(u) \geq E_V(\bar{u})$ and the "averaged" potential V in E_V conserves the properties of the initial potential W, in particular, $W \geq V \geq 0$ and V has the same wells as W (see Lemma 3). Then we prove that every transition layer $\bar{u}$ connecting two wells $u^{\pm}$ has the energy $E_V(\bar{u})$ bounded from below by a geodesic pseudo-distance geod_V between the wells $u^{\pm}$ (see Lemma 5). As the euclidian distance in $\mathbb{R}^N$ is absolutely continuous with respect to geod_V (see Lemma 4), we will conclude that $\bar{u}$ admits limits at $\pm\infty$ given by two wells of W (see Lemma 6). Note that in Section 3, we will give a second proof of the claim $\bar{u}(\pm\infty) = u^{\pm}$ without using the geodesic pseudo-distance geod_V.

We first introduce the energy functional E (defined in (1.1)) restricted to appropriate subsets $A \subset \Omega$ (e.g., A can be a subset of the form $I \times \omega$ for an interval $I \subset \mathbb{R}$, or a section $\{x_1\} \times \omega$): for every map $u \in \dot{H}^1(A, \mathbb{R}^N)$, we set

$$E(u, A) := \int_A |\nabla u|^2 + W(u)\,\mathrm{d}x,$$

so that for $A = \Omega$, we have $E(u) = E(u, A)$. For any interval $I \subset \mathbb{R}$, the Jensen inequality yields

$$E(u, I\times\omega) = \int_I \int_\omega \big(|\partial_1 u|^2 + |\nabla' u|^2 + W(u)\big)\,\mathrm{d}x'\,\mathrm{d}x_1 \geq |\omega| \int_I \Big|\frac{\mathrm{d}}{\mathrm{d}x_1}\overline{u}(x_1)\Big|^2 + e(u(x_1,\cdot))\,\mathrm{d}x_1,$$

where $\nabla' = (\partial_2, \ldots, \partial_d)$, $\bar{u}$ is the x'-average of u given in (1.2) and the x'-average energy e is defined by

$$e(v) := \fint_\omega \big(|\nabla' v|^2 + W(v)\big)\,\mathrm{d}x' \quad \text{ for all } v \in H^1(\omega, \mathbb{R}^N).$$

Introducing the "averaged" potential $V : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ defined for all $z \in \mathbb{R}^N$ by

$$V(z) := \inf\left\{ e(v) \ : \ v \in H^1(\omega, \mathbb{R}^N), \ \fint_\omega v\,\mathrm{d}x' = z \right\} \geq 0, \tag{2.1}$$

we have

$$E(u, I\times\omega) \geq |\omega| \int_I \left(\Big|\frac{\mathrm{d}}{\mathrm{d}x_1}\overline{u}(x_1)\Big|^2 + V(\overline{u}(x_1)) \right) \mathrm{d}x_1. \tag{2.2}$$

This observation is the starting point in the proof of the following lemma:

Lemma 3. *Let $W : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ be a lower semicontinuous function satisfying* **(H2)**. *Then the "averaged" potential $V : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ defined in* (2.1) *satisfies the following:*

1. *V is lower semicontinuous in $\mathbb{R}^N$,*
2. *for all $z \in \mathbb{R}^N$, $V(z) \leq W(z)$, the infimum in* (2.1) *is achieved and*[6] $\Big[V(z) = 0 \Leftrightarrow W(z) = 0\Big]$,
3. *$V_\infty := \liminf_{|z|\to\infty} V(z) > 0$,*
4. *for all interval $I \subset \mathbb{R}$ and for all $u \in \dot{H}^1(I \times \omega, \mathbb{R}^N)$, one has*

$$\frac{1}{|\omega|}E(u, I\times\omega) \geq E_V(\overline{u}, I), \quad E_V(\overline{u}, I) := \int_I \Big|\frac{\mathrm{d}}{\mathrm{d}x_1}\overline{u}(x_1)\Big|^2 + V(\overline{u}(x_1))\,\mathrm{d}x_1.$$

The new energy $E_V(\bar{u}) := E_V(\overline{u}, \mathbb{R})$ associated to the x'-average $\bar{u}$ will play an important role for proving the existence of the two limits $\overline{u}(\pm\infty)$.

[6] In particular, if W satisfies **(H1)**, then V satisfies **(H1)**, too.

Proof of Lemma 3. The claim *4* follows from (2.2). We divide the rest of the proof in three steps.

STEP 1: PROOF OF CLAIM *2*. Clearly, for all $z \in \mathbb{R}^N$, one has $V(z) \le e(z) = W(z)$. By the compact embedding $H^1(\omega) \hookrightarrow L^1(\omega)$, the lower semicontinuity of W, Fatou's lemma and the lower semicontinuity of the L^2 norm in weak L^2-topology (see [9]), we deduce that e is lower semicontinuous in weak $H^1(\omega, \mathbb{R}^N)$-topology. Then the direct method in the calculus of variations implies that the infimum is achieved in (2.1) (infimum that could be equal to $+\infty$ as W can take the value $+\infty$).

If $W(z) = 0$, then $V(z) = 0$ (as $V \le W$ in $\mathbb{R}^N$). Conversely, if $V(z) = 0$ with $z \in \mathbb{R}^N$, then a minimizer $v \in H^1(\omega, \mathbb{R}^N)$ in (2.1) satisfies $V(z) = e(v) = 0$ so that $v \equiv z$ and $W(z) = 0$.

STEP 2: V IS LOWER SEMICONTINUOUS IN $\mathbb{R}^N$. Let $(z_n)_{n\ge1}$ be a sequence converging to z in $\mathbb{R}^N$. We need to show that

$$V(z) \le \liminf_{n\to\infty} V(z_n).$$

Without loss of generality, one can assume that $(V(z_n))_{n\ge1}$ is a bounded sequence that converges to $\liminf_{n\to\infty} V(z_n)$. By Step 1, for each $n \ge 0$, there exists $v_n \in H^1(\omega, \mathbb{R}^N)$ such that

$$\fint_\omega v_n \, \mathrm{d}x' = z_n \quad \text{and} \quad e(v_n) = V(z_n).$$

Since $(z_n)_{n\ge1}$ and $(e(v_n))_{n\ge1}$ are bounded, $(v_n)_{n\ge1}$ is bounded in $H^1(\omega, \mathbb{R}^N)$ by the Poincaré-Wirtinger inequality. Thus, up to extraction, one can assume that $(v_n)_{n\ge1}$ converges weakly in H^1, strongly in L^1 and a.e. in ω to a limit $v \in H^1(\omega, \mathbb{R}^N)$. In particular, $\fint_\omega v \, \mathrm{d}x' = z$. Since e is lower semicontinuous in weak $H^1(\omega, \mathbb{R}^N)$-topology (by Step 1), we conclude

$$V(z) \le e(v) \le \liminf_{n\to\infty} e(v_n) = \liminf_{n\to\infty} V(z_n).$$

STEP 3: PROOF OF CLAIM *3*. Assume by contradiction that there exists a sequence $(z_n)_{n\ge1} \subset \mathbb{R}^N$ such that $|z_n| \to \infty$ and $V(z_n) \to 0$ as $n \to \infty$. Then, there exists a sequence of maps $(w_n)_{n\ge1}$ in $H^1(\omega, \mathbb{R}^N)$ satisfying

$$\int_\omega w_n(x') \, \mathrm{d}x' = 0 \quad \text{for each } n \in \mathbb{N} \quad \text{and} \quad e(z_n + w_n) \underset{n\to\infty}{\longrightarrow} 0.$$

By the Poincaré-Wirtinger inequality, we have that $(w_n)_{n\ge1}$ is bounded in H^1. Thus, up to extraction, one can assume that it converges weakly in H^1, strongly in L^1 and a.e. to a map $w \in H^1(\omega, \mathbb{R}^N)$. We claim that w is constant since

$$\fint_\omega |\nabla' w|^2 \, \mathrm{d}x' \le \liminf_{n\to\infty} \fint_\omega |\nabla' w_n|^2 \, \mathrm{d}x' \le \liminf_{n\to\infty} e(z_n + w_n) = 0.$$

We deduce $w \equiv 0$ since $\int_\omega w = \lim_{n\to\infty} \int_\omega w_n = 0$. Thus $w_n \to 0$ a.e and **(H2)** implies that for a.e. $x \in \omega$,

$$\liminf_{n\to\infty} W(z_n + w_n(x)) \ge \liminf_{|z|\to\infty} W(z) > 0,$$

which contradicts the fact that $e(z_n + w_n) \to 0$. □

For every lower semicontinuous function $W : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ satisfying **(H1)** and **(H2)**, we introduce the geodesic pseudo-distance geod_W in $\mathbb{R}^N$ endowed with the singular pseudo-metric $4W g_0$, g_0 being the standard euclidean metric in $\mathbb{R}^N$; this geodesic pseudo-distance (that can take the value $+\infty$) is defined for every $x, y \in \mathbb{R}^N$ by

$$\operatorname{geod}_W(x,y) := \inf\left\{ \int_{-1}^{1} 2\sqrt{W(\sigma(t))}|\dot\sigma|(t)\,\mathrm{d}t \ :\ \sigma\in \mathrm{Lip}_{ploc}([-1,1],\mathbb{R}^N),\ \sigma(-1)=x,\ \sigma(1)=y\right\}, \tag{2.3}$$

where $\mathrm{Lip}_{ploc}([-1,1],\mathbb{R}^N)$ is the set of continuous and **piecewise locally Lipschitz** curves[7] on $[-1,1]$:

$$\mathrm{Lip}_{ploc}([-1,1],\mathbb{R}^N) := \Big\{\sigma\in\mathcal{C}^0([-1,1],\mathbb{R}^N)\ : \text{there is a partition } -1=t_1<\cdots<t_{k+1}=1, \\ \text{with } \sigma\in \mathrm{Lip}_{loc}((t_i,t_{i+1})) \text{ for every } 1\le i\le k\Big\}.$$

By *pseudo-distance*, we mean that geod_W satisfies all the axioms of a distance; the only difference with respect to the standard definition is that a pseudo-distance can take the value $+\infty$. We will prove that geod_W yields a lower bound for the energy E (see Lemma 5); this plays an important role in the proof of our claim $\overline{u}(\pm\infty)=u^\pm$.

We start by proving some elementary facts about the pseudo-metric structure induced by geod_W on $\mathbb{R}^N$:

Lemma 4. *Let $W:\mathbb{R}^N\to\mathbb{R}_+\cup\{+\infty\}$ be a lower semicontinuous function satisfying* **(H1)** *and* **(H2)***. Then the function $\operatorname{geod}_W:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}_+\cup\{+\infty\}$ defines a pseudo-distance over $\mathbb{R}^N$ and the euclidian distance is absolutely uniform with respect to geod_W, i.e., for every $\delta>0$, there exists $\varepsilon>0$ such that for every $x,y\in\mathbb{R}^N$ with $\operatorname{geod}_W(x,y)<\varepsilon$, we have $|x-y|<\delta$.*

Proof of Lemma 4. In proving that $\operatorname{geod}_W:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}_+\cup\{+\infty\}$ defines a pseudo-distance over $\mathbb{R}^N$, the only non-trivial axiom to check is the non-degeneracy, i.e., $\operatorname{geod}_W(x,y)>0$ whenever $x\neq y$. In fact, we prove the stronger property that for every $\delta>0$, there exists $\varepsilon>0$ such that for every $x,y\in\mathbb{R}^N$, $|x-y|\ge\delta$ implies $\operatorname{geod}_W(x,y)\ge\varepsilon$ which also yields the absolute continuity of the euclidian distance with respect to geod_W. For that, we recall that the set $\{W=0\}$ is finite (by **(H1)**); therefore, w.l.o.g. we can assume that $\delta>0$ is small enough so that the open balls $B(p,\delta/2)$, for $p\in\{W=0\}$, are disjoint. We consider the following disjoint union of balls

$$\Sigma_\delta := \bigsqcup_{p\in\{W=0\}} B(p,\frac{\delta}{4}),$$

the distance between each ball being larger than $\delta/2$. We now take two points $x,y\in\mathbb{R}^N$ with $|x-y|\ge\delta$. In order to obtain a lower bound on $\operatorname{geod}_W(x,y)$, we take an arbitrary continuous and piecewise locally Lipschitz curve $\sigma:[-1,1]\to\mathbb{R}^N$ such that $\sigma(-1)=x$ and $\sigma(1)=y$. As $|x-y|\ge\delta$, by connectedness, the image $\sigma([-1,1])$ cannot be contained in Σ_δ. Thus, there exists $t_0\in[-1,1]$ with $\sigma(t_0)\notin\Sigma_\delta$. It implies that $B(\sigma(t_0),\delta/8)\cap\Sigma_{\delta/2}=\emptyset$. Since $\sigma(1)=y\in\Sigma_{\delta/2}$, the (continuous) curve $\sigma\big|_{[t_0,1]}$ has to get out of the ball $B(\sigma(t_0),\delta/8)$; in particular,

$$\int_{-1}^{1} 2\sqrt{W(\sigma(t))}|\dot\sigma|(t)\,\mathrm{d}t \ge \frac{\delta}{8}\inf_{z\in B(\sigma(t_0),\delta/8)}\sqrt{W(z)} \ge \frac{\delta}{8}\inf_{z\in\mathbb{R}^N\setminus\Sigma_{\delta/2}}\sqrt{W(z)}.$$

Since W is lower semicontinuous and bounded from below at infinity (by **(H2)**), we deduce that W is bounded from below by a constant $c_\delta>0$ on $\mathbb{R}^N\setminus\Sigma_{\delta/2}$. Taking the infimum over curves $\sigma\in\mathrm{Lip}_{ploc}([-1,1],\mathbb{R}^N)$ connecting x to y, we thus get

$$\operatorname{geod}_W(x,y)\ge\frac{\delta\sqrt{c_\delta}}{8}>0.$$

□

[7]Here we need to consider piecewise locally Lipschitz curves σ because W is not assumed locally bounded. In the case of a locally bounded W, one can restrict to Lipschitz curves σ.

We now use a regularization argument to derive the following lower bound on the energy:

Lemma 5. *Let $W : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ be a lower semicontinuous function. Then, for every interval $I \subset \mathbb{R}$ and every map $\sigma \in \dot{H}^1(I, \mathbb{R}^N)$ having limits $\sigma(\inf I)$ and $\sigma(\sup I)$ at the endpoints of I, we have*

$$E_W(\sigma, I) = \int_I \Big(|\dot\sigma(t)|^2 + W(\sigma(t)) \Big)\,\mathrm{d}t \geq \mathrm{geod}_W\big(\sigma(\inf I), \sigma(\sup I)\big). \tag{2.4}$$

Proof of Lemma 5. W.l.o.g. we assume that I is open. Since $\dot{H}^1(I, \mathbb{R}^N) \subset W^{1,1}_{loc}(I, \mathbb{R}^N)$, we can define the arc-length $s : I \to J := s(I) \subset \mathbb{R}$ by

$$s(t) := \int_{t_0}^{t} |\dot\sigma|(t')\,\mathrm{d}t', \quad t \in I,$$

where $t_0 \in I$ is fixed. Thus s is a nondecreasing continuous function with $\dot s = |\dot\sigma|$ a.e. in I. Then the arc-length reparametrization of σ, i.e.

$$\tilde\sigma(s(t)) := \sigma(t), \quad t \in I,$$

is well-defined and provides a Lipschitz curve $\tilde\sigma : J \to \mathbb{R}^N$ with constant speed, i.e. $|\dot{\tilde\sigma}| = 1$ a.e., and such that $\tilde\sigma(\inf J) = \sigma(\inf I)$ and $\tilde\sigma(\sup J) = \sigma(\sup I)$. W.l.o.g. we may assume that σ is not constant. Then we consider an arbitrary function $\varphi \in \mathrm{Lip}_{loc}((-1,1), \mathrm{int}J)$ which is nondecreasing and surjective onto the interior of the interval J and we set

$$\gamma(t) := \tilde\sigma(\varphi(t)), \quad t \in (-1,1).$$

So γ is a locally Lipschitz map that is continuous on $[-1,1]$ as $\tilde\sigma$ admits limits at $\inf J$ and $\sup J$; thus, $\gamma \in \mathrm{Lip}_{ploc}([-1,1], \mathbb{R}^N)$. The changes of variable $s = \varphi(t)$, resp. $s = s(t)$, yield

$$\int_{-1}^{1} 2\sqrt{W(\gamma(t))}|\dot\gamma|(t)\,\mathrm{d}t = \int_J 2\sqrt{W(\tilde\sigma(s))}|\dot{\tilde\sigma}|(s)\,\mathrm{d}s = \int_I 2\sqrt{W(\sigma(t))}\,|\dot\sigma|(t)\,\mathrm{d}t.$$

Combined with $\gamma(-1) = \sigma(\inf I)$ and $\gamma(1) = \sigma(\sup I)$, the definition of geod_W and the Young inequality imply

$$E_W(\sigma, I) \geq \int_I 2\sqrt{W(\sigma(t))}\,|\dot\sigma|(t)\,\mathrm{d}t = \int_{-1}^{1} 2\sqrt{W(\gamma(t))}|\dot\gamma|(t)\,\mathrm{d}t \geq \mathrm{geod}_W\big(\sigma(\inf I), \sigma(\sup I)\big).$$

□

The first step of Theorem 1 stating that $\overline{u}(\pm\infty) = u^\pm$ is a consequence of the following lemma:

Lemma 6. *Let $W : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ be a lower semicontinuous function satisfying* **(H1)** *and* **(H2)**. *Then for every map $\sigma \in \dot{H}^1(\mathbb{R}, \mathbb{R}^N)$ such that $E_W(\sigma, \mathbb{R}) < +\infty$ with E_W defined at* (2.4), *there exist two wells u^-, $u^+ \in \{W = 0\}$ such that $\lim_{t\to\pm\infty} \sigma(t) = u^\pm$.*

Proof of Lemma 6. We use the fact that the bound $E_W(\sigma, \mathbb{R}) < +\infty$ yields a bound on the total variation of $\sigma : \mathbb{R} \to \mathbb{R}^N$ where $\mathbb{R}^N$ is endowed with the pseudo-metric geod_W. More precisely, for every sequence $t_1 < \cdots < t_k$ in $\mathbb{R}$, we have by Lemma 5:

$$\sum_{i=1}^{k} \mathrm{geod}_W(\sigma(t_{i+1}), \sigma(t_i)) \leq \sum_{i=1}^{k} E_W(\sigma, [t_i, t_{i+1}]) \leq E_W(\sigma, \mathbb{R}) < +\infty.$$

In particular, for every $\varepsilon > 0$, there exists $R > 0$ such that for all $t, s \in \mathbb{R}$ with $t, s \geq R$ or $t, s \leq -R$, one has $\mathrm{geod}_W(\sigma(t), \sigma(s)) < \varepsilon$. Since by Lemma 4, smallness of $\mathrm{geod}_W(x,y)$ implies smallness of $|x-y|$, we deduce that σ has a limit $u^\pm \in \mathbb{R}^N$ at $\pm\infty$. Since $W(\sigma(\cdot))$ is integrable in $\mathbb{R}$, we have furthermore that $W(u^\pm) = 0$. □

Now we can prove the convergence of the x'-average $\bar{u}$ at $\pm\infty$ stated in Theorem 1:

Proof of the convergence in x'-average in Theorem 1. By Lemma 3, we have $E_V(\overline{u}, \mathbb{R}) < +\infty$ for a lower semicontinuous function $V : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ satisfying **(H1)** and **(H2)**. By Lemma 6, we deduce that there exists $u^\pm \in \{V = 0\} = \{W = 0\}$ such that $\lim_{t\to\pm\infty} \overline{u}(t) = u^\pm$. □

The pointwise convergence of $u(x_1, \cdot)$ as $x_1 \to \pm\infty$ stated in Theorem 1 is proved in the following:

Proof of the pointwise convergence in Theorem 1. We prove that $u(x_1, \cdot)$ converges a.e. in ω to $u^\pm \in \{W = 0\}$ as $x_1 \to \pm\infty$, where $u^\pm$ are the limits $\bar{u}(\pm\infty)$ of the x'-average $\bar{u}$ proved above. For that, we have by Fubini's theorem:

$$E(u) \geq \int_\Omega |\partial_1 u|^2 + W(u)\, \mathrm{d}x \geq \int_\omega E_W(u(\cdot, x'), \mathbb{R})\, \mathrm{d}x'$$

with the usual notation

$$E_W(\sigma, \mathbb{R}) = \int_\mathbb{R} |\dot{\sigma}|^2 + W(\sigma)\, \mathrm{d}x_1, \quad \sigma \in \dot{H}^1(\mathbb{R}, \mathbb{R}^N).$$

As $E(u) < \infty$, we deduce that $E_W(u(\cdot, x'), \mathbb{R}) < \infty$ for a.e. $x' \in \omega$. By Lemma 6, we deduce that for a.e. $x' \in \omega$, there exist two wells $u^\pm(x')$ of W such that

$$\lim_{x_1\to\pm\infty} u(x_1, x') = u^\pm(x'). \tag{2.5}$$

By (1.4), as $\bar{u}(\pm\infty) = u^\pm$, we know that $\|u(R_n^\pm, \cdot) - u^\pm\|_{L^2(\omega,\mathbb{R}^N)} \to 0$ as $n \to \infty$ for two sequences $R_n^\pm \to \pm\infty$. Up to a subsequence, we deduce that $u(R_n^\pm, \cdot) \to u^\pm$ a.e. in ω as $n \to \infty$. By (2.5), we conclude that $u^\pm(x') = u^\pm$ for a.e. $x' \in \omega$. □

3 The L^2 convergence

In this section, we prove that $u(x_1, \cdot)$ converges in $L^2(\omega, \mathbb{R}^N)$ to $u^\pm$ as $x_1 \to \pm\infty$. The idea is to go beyond the averaging procedure in Section 2 and keep the full information given by the x'-average energy e introduced at Section 2 over the set $H^1(\omega, \mathbb{R}^N)$. More precisely, we extend e to the space $L^2(\omega, \mathbb{R}^N)$ as follows

$$e(v) = \begin{cases} \displaystyle\fint_\omega \Big(|\nabla' v|^2 + W(v)\Big)\, \mathrm{d}x' & \text{if } v \in H^1(\omega, \mathbb{R}^N), \\ +\infty & \text{if } v \in L^2(\omega, \mathbb{R}^N) \setminus H^1(\omega, \mathbb{R}^N). \end{cases} \tag{3.1}$$

In particular, we have for every $u \in \dot{H}^1(\Omega, \mathbb{R}^N)$,

$$E(u) = \int_\mathbb{R} \Big(\|\partial_1 u(x_1, \cdot)\|^2_{L^2(\omega,\mathbb{R}^N)} + |\omega| e(u(x_1, \cdot))\Big)\, \mathrm{d}x_1. \tag{3.2}$$

In the sequel, we will also need the following properties of the energy e:

Lemma 7. *If $W : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ be a lower semicontinuous function satisfying* **(H2)***, then*

1. *e is lower semicontinuous in $L^2(\omega, \mathbb{R}^N)$,*
2. *the sets of zeros of e and W coincide; moreover $\Sigma := \{e = 0\} = \{W = 0\} \subset \mathbb{R}^N$ is compact,*

3. for every $\varepsilon > 0$, we have

$$k_\varepsilon := \inf \left\{ e(v) \; : \; v \in L^2(\omega, \mathbb{R}^N) \text{ with } d_{L^2}(v, \Sigma) \geq \varepsilon \right\} > 0.$$

Proof. We divide the proof in several steps:

STEP 1. LOWER SEMICONTINUITY OF e IN $L^2(\omega, \mathbb{R}^N)$. Indeed, let $v_n \to v$ in $L^2(\omega, \mathbb{R}^N)$. W.l.o.g., we may assume that $(e(v_n))_n$ is bounded, in particular, $(v_n)_n$ is bounded in $H^1(\omega, \mathbb{R}^N)$; thus, $(v_n)_n$ converges to v weakly in $H^1(\omega, \mathbb{R}^N)$. By Step 1 in the proof of Lemma 3, we know that $e\big|_{H^1(\omega,\mathbb{R}^N)}$ is lower semicontinuous w.r.t. the weak H^1 topology and the conclusion follows.

STEP 2. ZEROS OF e. The equality of the zero sets of e and W is straightforward thanks to the connectedness of ω. Thanks to the assumption **(H2)**, the set of zeros Σ of W is bounded and by the lower semicontinuity and non-negativity of W, the set of zeros Σ of W is closed; thus, Σ is compact in $\mathbb{R}^N$.

STEP 3. WE PROVE THAT $k_\varepsilon > 0$. Assume by contradiction that $k_\varepsilon = 0$ for some $\varepsilon > 0$. Then there exists a minimizing sequence $v_n \in L^2(\omega, \mathbb{R}^N)$ such that $d_{L^2}(v_n, \Sigma) \geq \varepsilon$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} e(v_n) = 0$. W.l.o.g., we may assume that $v_n \in H^1(\omega, \mathbb{R}^N)$ for every n as $\|v_n\|_{\dot{H}^1} \to 0$. Denoting $\overline{v_n}$ the (x'-)average of v_n, the Poincaré-Wirtinger inequality implies that the sequence $(w_n := v_n - \overline{v_n})_n$ converges in $H^1(\omega, \mathbb{R}^N)$ to 0. Up to extracting a subsequence, we may assume that $w_n \to 0$ for a.e. $x' \in \omega$.

Claim: The sequence $(\overline{v_n})_n$ is bounded in $\mathbb{R}^N$.

Indeed, assume by contradiction that there exists a subsequence of $(\overline{v_n})_n$ (still denoted by $(\overline{v_n})_n$) such that $|\overline{v_n}| \to \infty$ as $n \to \infty$. As W is l.s.c. and $w_n \to 0$ for a.e. $x' \in \omega$, the assumption (**H2**) implies

$$\liminf_{n\to\infty} W(v_n(x')) = \liminf_{n\to\infty} W(w_n(x') + \overline{v_n}) \geq \liminf_{|z|\to\infty} W(z) > 0 \quad \text{for a.e. } x' \in \omega$$

which by integration over $x' \in \omega$ contradicts the assumption $e(v_n) \to 0$. This finishes the proof of the claim.

As a consequence of the claim, we deduce that $(v_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\omega, \mathbb{R}^N)$. In particular, $(v_n)_{n\in\mathbb{N}}$ has a subsequence that converges in $L^2(\omega, \mathbb{R}^N)$ to a map $v \in H^1(\omega, \mathbb{R}^N)$ and we deduce $d_{L^2}(v, \Sigma) \geq \varepsilon$, in particular, v is not a zero of e, i.e., $e(v) > 0$. As e is l.s.c. in $L^2(\omega, \mathbb{R}^N)$, we have $0 = \lim_{n\to\infty} e(v_n) \geq e(v)$, which contradicts that $e(v) > 0$. □

Now we prove the L^2-convergence of $u(x_1, \cdot)$ to $u^\pm$ as $x_1 \to \pm\infty$:

Proof of the L^2-convergence in Theorem 1. Take $u \in H^1_{loc}(\Omega, \mathbb{R}^N)$ such that $E(u) < +\infty$ and set $\sigma(t) := u(t, \cdot) \in L^2(\omega, \mathbb{R}^N)$ for a.e. $t \in \mathbb{R}$. We prove that $\sigma(t)$ converges in $L^2(\omega, \mathbb{R}^N)$ to a limit that is a zero in Σ as $t \to +\infty$ (the proof of the convergence as $t \to -\infty$ is similar). Moreover, we will see that these limits are in fact the zeros $u^\pm$ of W given by the x'-average $\bar{u}$ and a.e. convergence of $u(x_1, \cdot)$ as $x_1 \to \pm\infty$.

STEP 1: CONTINUITY. We prove that $t \in \mathbb{R} \mapsto \sigma(t) \in L^2(\omega, \mathbb{R}^N)$ is continuous in $\mathbb{R}$, and moreover, it is a $\frac{1}{2}$-Hölder map. Indeed, for a.e. $t, s \in \mathbb{R}$, we have

$$d_{L^2}(\sigma(t), \sigma(s))^2 = \int_\omega \Big| \int_t^s \partial_{x_1} u(x_1, x')\, \mathrm{d}x_1 \Big|^2 \mathrm{d}x' \leq |t - s| \|\partial_{x_1} u\|^2_{L^2(\Omega, \mathbb{R}^N)}.$$

STEP 2: CONVERGENCE OF A SUBSEQUENCE $(\sigma(t_n))_n$ TO SOME $u^+ \in \Sigma$. Since $e(\sigma(\cdot)) \in L^1(\mathbb{R})$ by (3.2), there is a sequence $(t_n)_{n\in\mathbb{N}} \to +\infty$ such that $\lim_{n\to\infty} e(\sigma(t_n)) = 0$. Exactly like in Step 3 in the proof of Lemma 7, we deduce that $(\sigma(t_n))_{n\in\mathbb{N}}$ has a subsequence that converges strongly in $L^2(\omega,\mathbb{R}^N)$ to some map $\sigma_\infty \in L^2(\omega,\mathbb{R}^N)$ (the assumption **(H2)** is essential here). Since e is l.s.c. in L^2 and $e \geq 0$ in L^2, we deduce that $e(\sigma_\infty) = 0$ and so, there exists $u^+ \in \Sigma$ such that $\sigma_\infty \equiv u^+$.

Step 3: Convergence to u^+ in L^2 as $t \to +\infty$. Assume by contradiction that $\sigma(t)$ does not converge in $L^2(\omega,\mathbb{R}^N)$ to u^+ as $t \to \infty$. Then there is a sequence $(s_n)_{n\in\mathbb{N}} \to +\infty$ such that $\varepsilon := \inf_{n\in\mathbb{N}} d_{L^2}(\sigma(s_n), u^+) > 0$. Now, by Step 1, the curve $t \in [s_n, +\infty) \mapsto \sigma(t) \in L^2(\omega,\mathbb{R}^N)$ is continuous. Moreover, it starts out of the L^2-ball centered at u^+ with radius $\frac{3\varepsilon}{4}$ and, by Step 2, it has to enter in the L^2-ball centered at u^+ with radius $\frac{\varepsilon}{4}$. Therefore, the curve $\sigma_{|(s_n,+\infty)}$ has to cross the ring $\mathcal{R} := B_{L^2}(u^+, \frac{3\varepsilon}{4}) \setminus B_{L^2}(u^+, \frac{\varepsilon}{4})$, so it has L^2-length larger than $\frac{\varepsilon}{2}$, i.e.,

$$\int_{\{t\in(s_n,+\infty)\,:\,\sigma(t)\in\mathcal{R}\}} \|\partial_{x_1} u(t,\cdot)\|_{L^2(\omega,\mathbb{R}^N)}\,\mathrm{d}t = \int_{\{t\in(s_n,+\infty)\,:\,\sigma(t)\in\mathcal{R}\}} |\dot\sigma|_{L^2(\omega,\mathbb{R}^N)}\,\mathrm{d}t \geq \frac{\varepsilon}{2}.$$

Moreover, by the third claim in Lemma 7, we know that $e(\sigma(t)) \geq k_{\varepsilon/4}$ if $\sigma(t) \in \mathcal{R}$ (up to lowering ε, we may assume that the other zeros of Σ are placed at distance larger than 2ε from u^+, the assumption **(H1)** is essential here). We obtain

$$\begin{aligned}\int_{s_n}^{+\infty} \sqrt{e(u(t,\cdot))}\,\|\partial_{x_1} u(t,\cdot)\|_{L^2(\omega,\mathbb{R}^N)}\,\mathrm{d}t &\geq \int_{\{t\in(s_n,+\infty)\,:\,\sigma(t)\in\mathcal{R}\}} \sqrt{e(u(t,\cdot))}\,\|\partial_{x_1} u(t,\cdot)\|_{L^2(\omega,\mathbb{R}^N)}\,\mathrm{d}t\\ &\geq \frac{\varepsilon}{2}\sqrt{k_{\varepsilon/4}}.\end{aligned}$$

This is a contradiction with the assumption $E(u) < +\infty$ implying by (3.2):

$$\begin{aligned}2|\omega|^{\frac12} \int_{s_n}^{+\infty} \sqrt{e(u(t,\cdot))}\,\|\partial_{x_1} u(t,\cdot)\|_{L^2(\omega,\mathbb{R}^N)}\,\mathrm{d}t &\leq \int_{s_n}^{+\infty} \Big(|\omega| e(u(t,\cdot)) + \|\partial_{x_1} u(t,\cdot)\|^2_{L^2(\omega,\mathbb{R}^N)}\Big)\,\mathrm{d}t\\ &\underset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

Step 4: The L^2 limits $u^\pm$ coincide with the average limits $\bar u(\pm\infty)$. This is clear as L^2 convergence implies convergence in average. □

Remark 8. The above proof does not use (so, it is independent of) the almost everywhere convergence of $u(x_1,\cdot)$ as $x_1 \to \pm\infty$ or the convergence of the x'-average $\bar u$. Therefore, thanks to this proof, one can obtain as a consequence the almost everywhere convergence of $u(x_1,\cdot)$ as $x_1 \to \pm\infty$ as well as the convergence of the x'-average $\bar u$.

The above argument can also be used directly to obtain a second proof for the existence of limits of $\bar u$ at $\pm\infty$ without using the geodesic pseudo-distance geod_W (as presented in the proof in Section 2). For completeness, we redo the proof in the sequel:

Second proof of the convergence in x'-average in Theorem 1. Let $u \in \dot H^1(\Omega,\mathbb{R}^N)$ such that $E(u) < \infty$. We want to prove that the x'-average $\bar u$ admits a limit u^+ as $x_1 \to \infty$ and $W(u^+) = 0$ (the proof of the convergence as $x_1 \to -\infty$ is similar). Let V and E_V given by Lemma 3. Recall that $\Sigma := \{V = 0\} = \{W = 0\}$ and $E_V(\bar u) \leq \frac{1}{|\omega|} E(u) < \infty$.

Step 1. We prove that for every $\varepsilon > 0$,

$$\kappa_\varepsilon := \inf\big\{V(z) \;:\; d_{\mathbb{R}^N}(z,\Sigma) \geq \varepsilon\big\} > 0.$$

Assume by contradiction that there exists a sequence $(z_n)_n$ such that $V(z_n) \to 0$ and $d_{\mathbb{R}^N}(z_n,\Sigma) \geq \varepsilon$. By the third claim in Lemma 3, we deduce that $(z_n)_n$ is bounded, so that, up to a subsequence,

$z_n \to z$ for some $z \in \mathbb{R}^N$ yielding $d_{\mathbb{R}^N}(z, \Sigma) \geq \varepsilon$ and $V(z) = 0$, i.e., $z \in \Sigma$ (since V is l.s.c. and $V \geq 0$) which is a contradiction.

Step 2. There exists a sequence $(\bar{u}(t_n))_n$ converging to a well $u^+ \in \Sigma$. Indeed, as $V(\bar{u}) \in L^1(\mathbb{R})$, there exists a sequence $t_n \to \infty$ with $V(\bar{u}(t_n)) \to 0$. By **(H2)**, $(\bar{u}(t_n))_n$ is bounded, so that up to a subsequence, $\bar{u}(t_n) \to u^+$ as $t_n \to \infty$ for some point $u^+ \in \mathbb{R}^N$. As V is l.s.c. and $V \geq 0$, we deduce that $V(u^+) = 0$, i.e., $u^+ \in \Sigma$.

Step 3: Convergence of $\bar{u}$ to u^+ as $x_1 \to +\infty$. Assume by contradiction that $\bar{u}(x_1)$ does not converge to u^+ as $x_1 \to \infty$. Then there is a sequence $(s_n)_{n\in\mathbb{N}} \to +\infty$ such that $\varepsilon := \inf_{n\in\mathbb{N}} d_{\mathbb{R}^N}(\bar{u}(s_n), u^+) > 0$. As $\bar{u} : [s_n, +\infty) \to \mathbb{R}^N$ is continuous, by Step 2, it has to get out of the ball $B(\bar{u}(s_n), \varepsilon/4)$ and it has to enter in the ball $B(u^+, \varepsilon/4)$. Therefore, $\bar{u}$ has to cross the ring $\mathcal{R} := B(u^+, \frac{3\varepsilon}{4}) \setminus B(u^+, \frac{\varepsilon}{4})$. Moreover, by Step 1, we know that $V(\bar{u}(x_1)) \geq \kappa_{\varepsilon/4}$ if $\bar{u}(x_1) \in \mathcal{R}$ (where we assumed w.l.o.g. that $\varepsilon > 0$ is small enough so that the other zeros of Σ are placed at distance larger than 2ε from u^+). We obtain

$$\int_{s_n}^{+\infty} \sqrt{V(\bar{u}(x_1))} \, \Big|\frac{\mathrm{d}}{\mathrm{d}x_1}\bar{u}(x_1)\Big| \, \mathrm{d}x_1 \geq \int_{\{x_1 \in (s_n, +\infty) \,:\, \bar{u}(x_1) \in \mathcal{R}\}} \sqrt{V(\bar{u}(x_1))} \, \Big|\frac{\mathrm{d}}{\mathrm{d}x_1}\bar{u}(x_1)\Big| \, \mathrm{d}x_1 \geq \frac{\varepsilon}{2}\sqrt{\kappa_{\varepsilon/4}}.$$

This is a contradiction with the assumption $E_V(\bar{u}) < +\infty$ implying

$$2\int_{s_n}^{+\infty} \sqrt{V(\bar{u}(x_1))} \, \Big|\frac{\mathrm{d}}{\mathrm{d}x_1}\bar{u}(x_1)\Big| \, \mathrm{d}x_1 \leq \int_{s_n}^{+\infty} \Big(\Big|\frac{\mathrm{d}}{\mathrm{d}x_1}\bar{u}(x_1)\Big|^2 + V(\bar{u}(x_1))\Big) \, \mathrm{d}x_1 \underset{n\to\infty}{\longrightarrow} 0.$$

□

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