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par

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Approximations elliptiques d'énergies singulières sous contrainte de divergence

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Résumé

Cette thèse est consacrée à l'étude de certains problèmes variationnels de type transition de phase vectorielle ou "phase-field" qui font intervenir une contrainte de divergence. Ces modèles sont généralement basés sur une énergie dépendant d'un paramètre qui peut représenter une grandeur physique négligeable ou qui est liée à une méthode d'approximation numérique par exemple. Une question centrale concerne alors le comportement asymptotique de ces énergies et des minimiseurs globaux ou locaux lorsque ce paramètre tend vers 0. Cette thèse présente différentes stratégies prenant en compte la contrainte de divergence. Elles seront illustrées à travers l'étude de deux modèles. Le premier est une approximation du modèle Eulérien pour le transport branché par un modèle de type phase-field avec divergence prescrite. Nous montrons comment une estimation uniforme de l'énergie, en fonction de la contrainte sur la divergence, permet d'établir un résultat de Γ-convergence. Le second modèle, en lien avec la théorie du micromagnétisme, concerne des énergies de type Aviles-Giga dans un cadre vectoriel avec contrainte de divergence. Nous illustrerons dans quelle mesure la méthode d'entropie permet de caractériser les minimiseurs globaux. Dans certaines situations nous montrerons une conjecture de type De Giorgi concernant la symétrie $1D$ des minimiseurs globaux de l'énergie sous contrainte au bord.

Mots-clefs : Calcul des Variations, Γ-convergence, Problèmes à discontinuité libre, Transition de phase, Ginzburg-Landau, Transport branché.

ELLIPTIC APPROXIMATIONS OF SINGULAR ENERGIES UNDER DIVERGENCE CONSTRAINT

Abstract

This thesis is devoted to the study of phase-field type variational models with divergence constraint. These models typically involve an energy depending on a parameter which represents a negligible physical quantity or is linked to some numerical approximation method for instance. A central question concerns the asymptotic behavior of these energies and of their global or local minimizers when this parameter goes to 0. We present different strategies which allow to take the divergence constraint into account. They will be illustrated in two models. The first one is a phase-field type approximation, involving a divergence constraint, of the Eulerian model for branched transportation. We illustrate how uniform estimates on the energy, depending on the constraint on the divergence, allow to establish a Γ-convergence result. The second model, related to micromagnetics, concerns Aviles-Giga type energies for divergence-free vector fields. We use the entropy method in order to characterize global minimizers. In some situations, we will prove a De Giorgi type conjecture concerning the one-dimensional symmetry of global minimizers under boundary condition.

Keywords : Calculus of Variations, Γ-convergence, Free discontinuity problems, Phase transition, Ginzburg-Landau, Branched transportation.

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Introduction générale

Problèmes variationnels dépendant d'un paramètre

Le calcul des variations regorge d'exemples de problèmes, issus de la physique ou motivés par des applications théoriques, où la fonctionnelle à minimiser dépend d'un paramètre. Celui-ci peut représenter une grandeur physique, géométrique ou encore un paramètre de discrétisation. Dans de nombreux exemples, ces problèmes deviennent singuliers et leur étude très complexe quand ce paramètre se rapproche de certaines valeurs critiques. Cependant, l'étude asymptotique de ces modèles lorsque le paramètre varie révèle souvent des problèmes variationnels plus simples, indépendants du paramètre, et qui permettent de mieux cerner l'essence du problème initial. Par exemple, le modèle scalaire et non convexe de Modica-Mortola [48] fait intervenir le problème purement géométrique correspondant à la minimisation du périmètre.

Ce type de problèmes dépendant d'un paramètre, disons $\varepsilon > 0$, qui dans notre cas sera destiné à tendre vers 0, s'exprime généralement comme un problème de minimisation de la forme

$$\min\{E_\varepsilon(u) \;:\; u \in K_\varepsilon\}, \tag{0.0.1}$$

où les fonctionnelles E_ε sont définies sur un ensemble fonctionnel, et K_ε représente une contrainte éventuellement dépendante de ε. Dans cette thèse, nous sommes tout particulièrement intéressés au cadre vectoriel où K_ε représente une contrainte sur la divergence.

Dans les exemples que nous rencontrerons, on s'attend à ce que le problème de minimisation (0.0.1), dans le régime asymptotique lorsque ε converge vers 0, devienne un problème variationnel de la forme

$$\min\{E(u) \;:\; u \in K\}, \tag{0.0.2}$$

K étant la contrainte asymptotique sur les structures admissibles pour le problème limite. Nous rencontrerons des exemples, notamment en micromagnétisme, où l'étude du problème limite (0.0.2) est plus simple et plus instructive que celle du problème dépendant d'un paramètre (0.0.1). Parfois, à l'inverse, on se propose d'étudier un problème singulier de la forme (0.0.2) et il est utile, à des fins aussi bien théoriques que numériques, de les approcher par des problèmes plus réguliers du type (0.0.1). Une telle approximation peut être obtenue par perturbation au moyen d'un terme "elliptique", c'est ce que nous verrons dans certains problèmes de transport. Un outil mathématique fondamental permettant de donner un cadre précis à l'analyse asymptotique de ce type de problèmes est la théorie de la Γ-convergence.

Γ-convergence

La théorie de la Γ-convergence, introduite par E. De Giorgi [25], est un type de convergence sur les fonctionnelles dépendant d'un paramètre qui garantit la convergence des minimiseurs vers un minimiseur de la fonctionnelle limite ainsi que celle des valeurs minimales. Pour une étude apprionfondie de cette théorie, le lecteur pourra se référer à [18] ou [24]. Nous nous contentons ici d'en donner les principaux aspects et outils utiles pour notre étude.

Définition 0.0.1. Soit (X, d) un espace métrique et $(F_\varepsilon)_{\varepsilon>0} : X \to \mathbb{R} \cup \{+\infty\}$ une suite de fonctions définies sur X. On dit que la suite $(F_\varepsilon)_{\varepsilon>0}$ Γ-converge vers $F : X \to \mathbb{R} \cup \{+\infty\}$ et on note $F = \Gamma - \lim\limits_{\varepsilon\to 0} F_\varepsilon$ si les deux propriétés suivantes sont vérifiées :

Borne inférieure Pour toute suite $(x_\varepsilon)_{\varepsilon>0} \subset X$ qui converge vers $x \in X$ lorsque $\varepsilon \to 0$,

$$F(x) \leq \liminf_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) \quad . \tag{0.0.3}$$

Borne supérieure Pour tout $x \in X$, il existe une suite $(x_\varepsilon)_{\varepsilon>0}$ qui converge vers x lorsque $\varepsilon \to 0$ et vérifiant la condition suivante,

$$F(x) = \lim_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) \quad . \tag{0.0.4}$$

En d'autres termes, cela signifie que l'inégalité (0.0.3) ne peut pas être améliorée. Nous définissons également la $\Gamma - \liminf$ de la suite $(F_\varepsilon)_{\varepsilon>0}$ par

$$\Gamma - \liminf_{\varepsilon\to 0} F_\varepsilon(x) := \inf\{\liminf_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) \ : \ x_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} x\} \quad \text{pour tout } x \in X,$$

et la $\Gamma - \limsup$ de la suite $(F_\varepsilon)_{\varepsilon>0}$ par

$$\Gamma - \limsup_{\varepsilon\to 0} F_\varepsilon(x) := \inf\{\limsup_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) \ : \ x_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} x\} \quad \text{pout tout } x \in X.$$

Clairement, la suite $(F_\varepsilon)_{\varepsilon>0}$ Γ-converge vers F si et seulement si les égalités suivantes sont vérifiées :

$$F = \Gamma - \liminf_{\varepsilon\to 0} F_\varepsilon = \Gamma - \limsup_{\varepsilon\to 0} F_\varepsilon \quad .$$

Nous aurons également besoin de la notion de coercivité suivante : la suite $(F_\varepsilon)_{\varepsilon>0}$ est dite *equi-coercive* sur X si pour tout $R > 0$ il existe un ensemble compact $K \subset X$ tel que

$$\forall \varepsilon > 0, \{x \in X \ : \ F_\varepsilon(x) \leq R\} \subset K \quad .$$

Parmis toutes les propriétés de la Γ-convergence, nous retiendrons les suivantes :

Proposition 0.0.2. *Soit $(F_\varepsilon)_{\varepsilon>0}$ une suite de fonctionnelles Γ-convergent vers $F : X \to \mathbb{R} \cup \{+\infty\}$ lorsque $\varepsilon \to 0$.*

- **Semi-continuité de la Γ-limite :** *F est semi-continue inférieurement.*
- **Existence des minimiseurs :** *Supposons par ailleurs que les deux propriétés suivantes sont vérifiées :*

- **Compacité :** *toute suite d'énergie bornée* $(x_\varepsilon)_{\varepsilon>0} \subset X$*, i.e.*

$$\sup\{F_\varepsilon(x_\varepsilon) \ : \ \varepsilon > 0\} < +\infty, \tag{0.0.5}$$

est relativement compacte dans (X, d)*. C'est le cas, par exemple, si* $(F_\varepsilon)_{\varepsilon>0}$ *est equi-coercive.*

- **Finitude :** $\inf_X F > -\infty$.

Alors le minimum de F *est atteint et* $\min_X F = \lim_{\varepsilon\to 0} \inf_X F_\varepsilon$.

- **Stabilité des minimiseurs :** *Soit* $(x_\varepsilon)_{\varepsilon>0}$ *une suite de minimiseurs de* F_ε *et* x *une valeur d'adhérence de la suite* $(x_\varepsilon)_{\varepsilon>0}$*. Alors* x *est un minimiseur de* F*.*
- **Stabilité de la Γ-convergence :** *Pour toute fonctionnelle continue* $G : X \to \mathbb{R} \cup \{+\infty\}$*, la suite* $(F_\varepsilon + G)_{\varepsilon>0}$ Γ*-converge vers* $F + G$*.*

Quand à la motivation de la Γ-convergence, deux points de vue sont possibles. Soit la suite $(F_\varepsilon)_{\varepsilon>0}$ est donnée, par exemple issue d'un problème physique, et déterminer la Γ-limite de cette fonctionnelle permet de mieux comprendre le comportement des minimiseurs. Soit la fonctionnelle F est une fonctionnelle singulière donnée (par exemple l'énergie se concentre sur les ensembles de codimension un comme dans le modèle de Modica-Mortola), et F_ε intervient comme une régularisation de F permettant, par exemple, de mettre en place des méthodes numériques d'approximation. De telles fonctionnelles singulières F peuvent être motivées par une application théorique, telle que l'inégalité isopérimétrique, ou pratique, comme on le verra dans certains problèmes de transport.

Deux exemples scalaires de Γ-convergence

Le modèle de transition de phase de Modica-Mortola

Un exemple fondamental de Γ-convergence remonte aux travaux pionniers de L. Modica et S. Mortola [48] qui étudièrent la fonctionnelle définie de la manière suivante : pour tout $u \in L^1(\Omega)$ défini sur un ouvert borné $\Omega \subset \mathbb{R}^d$,

$$M_\varepsilon(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega \varepsilon|\nabla u|^2 + \frac{1}{\varepsilon}W(u) & \text{si } u \in H^1(\Omega), \\ +\infty & \text{sinon,} \end{cases}$$

où $W : \mathbb{R}^d \to \mathbb{R}^+$ est un potentiel s'annulant en deux points distincts, disons 0 et 1, appelés puits du potentiel. Ces deux puits peuvent correspondre à deux états possibles (appelées phases) pour un système composé de deux matières de nature différente (huile et eau) ou encore une même matière dans deux états différents (phases liquide et solide). Les valeurs intermédiaires, $0 < u < 1$, représentent alors un état transitoire caractérisé par le mélange des deux phases. Lorsque ε tend vers 0, on s'attend à ce qu'une suite de minimiseurs, ou même une suite $(u_\varepsilon)_{\varepsilon>0}$ d'énergie uniformément bornée, i.e. $\sup E_\varepsilon(u_\varepsilon) < +\infty$, se concentre sur les valeurs 0 et 1. En effet, pour de telles suites, on sait que

$W(u_\varepsilon)$ converge vers 0 dans $L^1(\Omega)$. Cela signifie en particulier que le volume de la "zone transitoire" converge vers 0. On observe en pratique, par exemple dans des simulations numériques, que pour ε très petit, u vaut 0 ou 1 dans deux grandes zones occupant presque tout le domaine Ω. La transition entre les deux phases est assurée sur une bande dont la largeur est d'ordre ε autour de l'interface, c'est à dire l'hypersurface située entre les zones occupées par les deux phases $\{u = 0\}$ et $\{u = 1\}$ (voir Figure 1). Au voisinage de chaque point sur cette interface, à l'échelle microscopique, u est décrit par un profil $1D$ optimal reliant les deux valeurs 0 et 1 (voir figure 2). Formellement, si $S \subset \Omega$ est l'hypersurface représentant l'interface, on s'attend à ce qu'un minimiseur u soit approché par la formule $u(x) \sim \varphi\left(\frac{\mathrm{dist}(x,S)}{\varepsilon}\right)$ au voisinage de S, où $\varphi : \mathbb{R} \to \mathbb{R}$ est le profil optimal $1D$, solution du problème variationnel suivant

$$\min\left\{\frac{1}{2}\int_{\mathbb{R}} |\varphi'(t)|^2 + W(\varphi)\,\mathrm{d}t \ : \ \varphi : \mathbb{R} \to \mathbb{R},\ \varphi(-\infty) = 0,\ \varphi(+\infty) = 1\right\}. \tag{0.0.6}$$

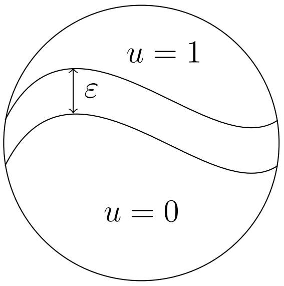

FIGURE 1 – Interface entre deux phases

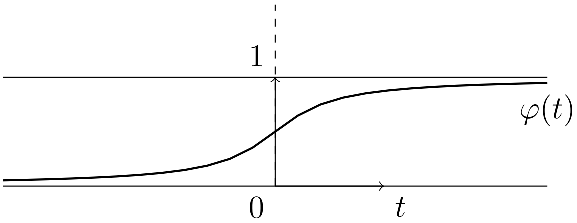

FIGURE 2 – Profil $1D$

Dans [48], les auteurs ont démontré le résultat de Γ-convergence suivant :

Théorème. *Soit $\Omega \subset \mathbb{R}^d$ un ouvert borné, Lipschitz et soit $W \in \mathbf{C}^0(\mathbb{R}^d, \mathbb{R}^+)$ un potentiel tel que $W(z) = 0 \Leftrightarrow z = 0$ ou 1. Alors la suite de fonctionnelles $(M_\varepsilon)_{\varepsilon>0}$ Γ-converge dans $L^1(\Omega)$ vers la fonctionnelle suivante :*

$$M_0(u) = \begin{cases} c_W \operatorname{Per}(S) & \text{si } u = \mathbf{1}_S \text{ pour un ensemble } S \subset \Omega \text{ de périmètre fini,} \\ +\infty & \text{sinon,} \end{cases}$$

où la constante c_W est donnée par la formule $c_W = \int_0^1 \sqrt{W(t)}\,\mathrm{d}t$, correspondant exactement à la valeur minimale du problème $1D$, (0.0.6).

Par ailleurs, la Γ-convergence a toujours lieu lorsqu'on ajoute une contrainte de volume de la forme $\int_\Omega u = V$, représentant la quantité présente dans le domaine Ω de la phase 1.

Théorème. *Soit $V \geq 0$ fixé et Ω, W vérifiant les mêmes hypothèses que dans le théorème précédent. La fonctionnelle $\overline{M}_\varepsilon$ définie sur $L^1(\Omega)$ par*

$$\overline{M}_\varepsilon(u) := \begin{cases} M_\varepsilon(u) & \text{si } \int_\Omega u = V, \\ +\infty & \text{sinon,} \end{cases}$$

Γ-converge dans $L^1(\Omega)$ lorsque $\varepsilon \to 0$ vers vers la fonctionnelle $\overline{M}_0$ définie par

$$\overline{M}_0(u) = \begin{cases} M_0(u) & \textit{si } \int_\Omega u = V, \\ +\infty & \textit{sinon.} \end{cases}$$

Ce résultat est très intéressant dans le sens où il fait surgir un problème géométrique, lié ici au problème isopérimétrique, à partir d'un problème scalaire défini sur $L^1(\Omega)$. Le sens des deux théorèmes précédents est que le seul comportement asymptotique perceptible à l'échelle macroscopique (celle du domaine Ω) est déterminé par un problème purement géométrique qui consiste en la minimisation de la surface de l'interface. D'un point de vue empirique, pour ε très petit, le problème de minimisation $\min\{M_\varepsilon(u) \ : \ \int u = V\}$ revient à d'abord minimiser la surface de l'interface (échelle macroscopique) puis à minimiser le profil optimal de u (échelle microscopique) pour la transition entre les deux phases (voir (0.0.6)).

L'ingrédient clef dans la preuve des deux théorèmes précédents est l'inégalité suivante, conséquence de l'inégalité de Young :

$$\frac{1}{2}\left(\varepsilon|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\right) \geq \sqrt{W(u)}|\nabla u| = |\nabla(F \circ u)|, \tag{0.0.7}$$

où F et une primitive de $\sqrt{W}$. En intégrant cette inégalité sur tout le domaine Ω, on obtient que la variation totale de $F \circ u$, $\mathrm{TV}(F \circ u)$, est controlée par l'énergie $M_\varepsilon(u)$. Observons que lorsque u est de la forme $u = \mathbf{1}_S$ pour un ensemble de périmètre fini $S \subset \Omega$, on obtient l'identité $\mathrm{TV}(F \circ u) = (F(1) - F(0)) \operatorname{Per}(S) = c_W \operatorname{Per}(S)$, essentielle afin d'obtenir la borne inférieure dans la Γ-convergence de M_ε vers M_0. Par ailleurs, notons que toutes les inégalités précédentes deviennent des égalités lorsque u est de la forme $u = \varphi(\frac{x \cdot \nu}{\varepsilon})$ où $\nu \in \mathbb{S}^{d-1}$ et φ est solution du problème (0.0.6).

Les théorèmes précédents reposent en particulier sur le fait que dans ce cadre scalaire, les profils $1D$ sont toujours optimaux. Nous verrons dans la deuxième partie de cette thèse et dans la dernière partie de cette introduction, des exemples vectoriels avec contrainte de divergence où cette propriété n'est plus vérifiée. Dans ce cadre, il s'avère que l'étude du profil optimal peut-être très délicate et des microstructures plus ou moins complexes peuvent apparaître. En fait, contrairement au cas de Modica-Mortola où des hypothèses génériques sont demandées pour le potentiel, l'optimalité du profil $1D$ nécéssite des hypothèses assez fortes sur W dans ce cadre vectoriel avec contrainte de divergence.

Une autre propriète, qui résulte également de l'estimation (0.0.7), concerne la compacité pour les fonctionnelles M_ε :

Théorème. *Soit $\Omega \subset \mathbb{R}^d$ un ouvert borné et Lipschitz. Soit $W \in \mathbf{C}^0(\mathbb{R}, \mathbb{R}^+)$ un potentiel à deux puits 0 et 1, i.e. $W(z) = 0 \Leftrightarrow z = 0$ ou 1, et à croissance sur-linéaire à l'infini :*

$$\exists L, R > 0, \, \forall z \in \mathbb{R}^d, \, |z| \geq R \Longrightarrow W(z) \geq L|z|.$$

La suite de fonctionnelle $(M_\varepsilon)_{\varepsilon>0}$ vérifie alors la propriété de compacité au sens de la Proposition 0.0.2 : toute suite d'énergie bornée, i.e. vérifiant (0.0.5), est relativement compacte dans $L^1(\Omega)$.

Nous faisons observer que les résultats théoriques précédents ont eu d'intéréssantes applications numériques, notamment dans [52] où E. Oudet utilise la Γ-convergence des fonctionnelles M_ε afin de mettre en œuvre une méthode numérique pour des problèmes d'interface.

Un modèle en dimension 1 avec un potentiel dégénéré à l'infini

Dans [15], G. Bouchitté, C. Dubs et P. Seppecher ont étudié la suite de fonctionnelles suivante, issues des modèles de Cahn-Hilliard pour l'équilibre des gouttelettes chargées en dimension 1 :

$$F_\varepsilon(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_I \varepsilon|u'(x)|^2 + \dfrac{1}{\varepsilon}W(u) & \text{si } u \in H^1(I,\mathbb{R}) \text{ et } u \geq 0 \text{ p.p.,} \\ +\infty & \text{sinon,} \end{cases}$$

où $I \subset \mathbb{R}$ est un intervalle ouvert et le potentiel $W : \mathbb{R}^+ \to \mathbb{R}^+$ vérifie les conditions suivantes :

- W est continu sur $\mathbb{R}^+$, $W(0) = 0$ et $W(t) > 0$ pour $t > 0$,
- $\liminf\limits_{t\to 0^+} \frac{W(t)}{t} > 0$,
- $\exists \beta < 1$, $c_\beta > 0$, $\lim\limits_{t\to\infty} \frac{W(t)}{t^\beta} = c_\beta$,
- W est croissante sur $[0, t_0]$ pour un $t_0 > 0$.

Un exemple de potentiel vérifiant ces hypothèses pour $\beta \in (0,1)$ est donné par $W(t) = t^\beta$. Lorsque $\beta \leq 0$ l'application $t \mapsto t^\beta$ est singulière en l'origine mais un exemple de potentiel W satisfaisant les hypothèses précédentes est facilement obtenu par une modification au voisinage de l'origine, par exemple $W(t) = \inf\{t^\beta, t\}$. L'énergie F_ε correspond dans un certain sens à une énergie de Modica-Mortola en dimension 1 pour un potentiel à deux puits W dont le premier puits est en $u = 0$ et dont le deuxième puits a été envoyé à l'infini. Notons que dans le cas $\beta < 0$, 0 et $+\infty$ sont réellement les deux puits du potentiel W prolongé par 0 en $u = +\infty$. Dans le cas $0 < \beta < 1$, bien que W ne s'annule pas en l'infini, on peut tout de même remarquer que, en considérant une contrainte de volume sur u correspondant au volume total des gouttelettes, le terme concave $\int_I W(u)$ favorise la concentration, c'est à dire les fonctions u s'annulant sur une grande partie du domaine et prenant de grandes valeurs sur une petite région, correspondant au support des gouttelettes. Dans cette direction, observons que si u vaut $t > 0$ sur un intervalle de longueur $L > 0$ et 0 ailleurs, et si u vérifie la contrainte de volume $Lt = \int u = V$, alors $\int W(u) = LW(t) = VW(t)/t$ qui tend vers 0 à l'infini grâce à l'hypothèse $W(t) \sim t^\beta$ avec $\beta < 1$ en l'infini. De cette façon, dans [15], les auteurs ont montré que dans ce type de modèle, l'énergie se concentrait sur un ensemble fini de points. Autrement dit, l'énergie F_ε se comporte asymptotiquement comme une énergie atomique de la forme

$$F(\mu) = \begin{cases} \displaystyle\sum_{i=1}^{k} f(m_i) & \text{si } \mu = \sum_{i=1}^k m_i\delta_{x_i} \text{ avec } m_i > 0,\ x_i \in \overline{I} \text{ distincts,} \\ +\infty & \text{si } \mu \text{ n'est pas atomique et positive,} \end{cases} \tag{0.0.8}$$

où f est une fonction positive définie sur $\mathbb{R}^+$ qui représente l'énergie d'une gouttelette en fonction de sa masse. Dans [15], les auteurs ont démontré la Γ-convergence de F_ε vers une fonctionnelle atomique de ce type, qui se concentre sur les atomes x_i correspondant aux centres du support des gouttelettes (ou une extrémité pour les points situés au bord de I). Il est à noter, cependant, que la fonction f, qui représente le coût d'une gouttelette, devrait aussi dépendre du point x_i puisque les gouttelettes du bord, $x_i \in \partial I$, ne réalisent la transition entre 0 et l'infini qu'une seule fois alors que les points intérieurs la réalisent deux fois et coûtent donc "plus cher" (voir figures 3).

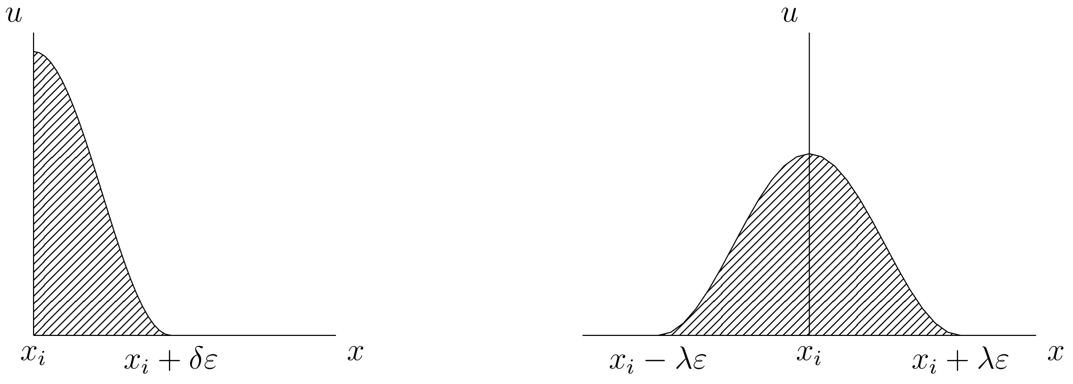
u
x_i
$x_i + \delta\varepsilon$
x
u
$x_i - \lambda\varepsilon$
x_i
$x_i + \lambda\varepsilon$
x

Figure 3 – Une gouttelette sur le bord (à gauche) et à l'intérieur (à droite)

Cadre vectoriel avec contrainte de divergence

Cette thèse est majoritairement consacrée à l'étude de certains modèles de type Modica-Mortola dans un cadre vectoriel avec contrainte de divergence. Comme nous l'avons déjà fait remarquer, plusieurs difficultés sont rencontrées à partir de la dimension 2. Celles-ci peuvent provenir de la nature de l'ensemble des puits, $\{z \in \mathbb{R}^d \ : \ W(z) = 0\}$, qui pourrait ne plus être discret ce qui fait une différence majeure par rapport au modèle scalaire de Modica-Mortola. Par exemple, dans le modèle d'Aviles-Giga, cet ensemble est le cercle unité ce qui rend son étude plus délicate puisque les structures limites sont plus complexes, en l'occurence, des champs de vecteurs unitaires à divergence nulle. De manière générale, tous les modèles que nous allons étudier ont la particularité de faire intervenir une contrainte sur la divergence. Celle-ci peut changer radicalement la nature des modèles ou encore leur étude mathématique. Comme nous l'avons vu précédemment, la contrainte de divergence peut être source de difficulté pour ce qui est de montrer la persistence de la borne supérieure (c'est à dire (0.0.4)) lorsqu'on ajoute la contrainte. Naturellement, l'étude le la borne inférieure (c'est à dire (0.0.3)) présente également de nouvelles difficultés en dimension supérieure puisque de nombreux outils et méthodes spécifiques à la dimension 1 ne s'appliquent plus. Par ailleurs, dans la deuxième partie, nous étudierons des modèles avec contrainte de divergence nulle, principalement en dimension 2, c'est à dire pour des champs gradients (à une rotation près). Ces modèles peuvent être considérés comme des modèles d'ordre supérieure. En ce sens, leur étude peut s'avérer plus délicate que celle de modèles d'ordre 1 comme le modèle de Modica-Mortola.

L'étude des modèles présentés dans cette introduction générale sera reprise et étoffée dans les introductions respectives de chaque partie de la thèse.

Champs gradients

Nous présentons ici des modèles en dimension 2 définis avec une contrainte de divergence nulle. Grâce au lemme de Poincaré, sur un domaine simplement connexe de $\mathbb{R}^2$, les champs de vecteurs u à divergence nulle s'écrivent comme le gradient orthogonal d'un potentiel (scalaire), $u = (\nabla\varphi)^\perp$, ce qui justifie la terminologie "champ gradient".

Modèle d'Aviles-Giga

Ce modèle a été introduit par P. Aviles et Y. Giga [6] et fortement étudié par la suite [7, 4, 39, 28, 27]. Cette énergie, en lien avec l'étude des cristaux liquides ou le micromagnétisme, est définie de la manière suivante : pour tout $u \in L^1(\Omega)$,

$$\mathrm{AG}_\varepsilon(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega \varepsilon|\nabla u(x)|^2 + \frac{1}{\varepsilon}(1-|u(x)|^2)^2 \,\mathrm{d}x & \text{si } u \in H^1(\Omega,\mathbb{R}^2) \text{ et } \nabla\cdot u = 0, \\ +\infty & \text{sinon,} \end{cases}$$

où $\Omega \subset \mathbb{R}^d$ est un ouvert borné. Il est assez fréquent d'ajouter une condition de Neumann au bord du domaine, $u\cdot n = 0$ sur $\partial\Omega$, ce qui revient, si Ω est régulier, à imposer que la divergence de $u\mathbf{1}_\Omega$ (c'est à dire u est prolongé par 0 en dehors de Ω) est nulle au sens des distributions. Cette contrainte, $\nabla\cdot(u\mathbf{1}_\Omega) = 0$, peut provenir de considérations physiques si l'on voie ce modèle comme modèle jouet pour l'étude du micromagnétisme (cf. partie suivante). Comme dans le modèle de Modica-Mortola, on s'attend à ce que les suites de minimiseurs ou même les suites d'énergie bornée prennent à la limite des valeurs sur le cercle unité $\mathbb{S}^1$. Alors que pour un potentiel W à deux puits, le coût du profil optimal de transition entre les deux phases est une constante c_W, dans la situation plus complexe d'Aviles-Giga, ce coût est représenté par une fonction définie sur les couples (u^-, u^+) où $u^\pm \in \mathbb{S}^1$. Étant donné que le potentiel considéré, $W(z) = (1-|z|^2)^2$, est invariant par rotation, on peut s'attendre à ce que cette fonction ne dépende en fait que de la distance entre les deux puits, $|u^+ - u^-|$. Autrement dit, l'énergie limite des fonctionnelles AG_ε lorsque $\varepsilon \to 0$, au sens de la Γ-convergence, devrait se concentrer sur la ligne de saut $J(u)$ des champs de vecteurs unitaires à divergence nulle. On aboutit alors à une énergie de la forme

$$E_c(u) = \begin{cases} \displaystyle\int_{J(u)} c(|u^+ - u^-|)\,\mathrm{d}\mathcal{H}^1 & \text{si } |u| = 1 \text{ p.p. et } \nabla\cdot u = 0, \\ +\infty & \text{sinon,} \end{cases}$$

où $u^\pm$ représentent les traces de u de part et d'autre de sa ligne de saut orientée par un vecteur normal $\nu \in \mathbb{S}^1$, et $c : \mathbb{R}^+ \to \mathbb{R}^+$ est la *fonction coût*. Notons que $u^\pm$ et $J(u)$ sont bien définis lorsque u est à variation bornée. La fonction coût c est facilement calculable à partir d'une analyse asymptotique $1D$ qui consiste à minimiser une énergie analogue à (0.0.6). Les énergies de cette forme (pour une fonction coût quelconque) sont généralement appelées *énergies de ligne.* Des travaux [7] et [39] resort le résultat suivant :

Théorème. *Pour toute suite* $(u_\varepsilon)_{\varepsilon>0} \subset L^1(\Omega, \mathbb{R}^2)$ *convergeant fortement vers* $u \in \mathrm{BV}(\Omega, \mathbb{R}^2)$ *dans* L^1*, on a*

$$E_c(u) \leq \liminf_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon),$$

où la fonction $c : \mathbb{R}^+ \to \mathbb{R}^+$ *correspond au coût cubique :*

$$c(t) = \frac{t^3}{6}.$$

De plus, E_c *est semi-continue inférieurement dans le sens suivant : pour toute suite* $(u_n)_{n\geq 1} \subset \mathrm{BV}(\Omega, \mathbb{R}^2)$ *qui converge fortement vers* $u \in \mathrm{BV}$ *dans* L^1*, l'inégalité suivante est vérifiée,*

$$E_c(u) \leq \liminf_{n \to \infty} E_c(u_n) \quad .$$

Dans le cas où $u \notin \mathrm{BV}$, il est possible de remplacer E_c, défini seulement sur $\mathrm{BV}(\Omega, \mathbb{R}^2)$, par sa relaxation $\overline{E}_c$ dans L^1 : $\overline{E}_c(u) = \inf\{\liminf_{n\to\infty} E_c(u_n) \quad : \quad u_n \to u \text{ dans } L^1\}$. $\overline{E}_c$ est alors semi-continue inférieurement sur L^1 et on a à la fois $\overline{E}_c(u) = E_c(u)$ et $\overline{E}_c(u) \leq \Gamma - \liminf_{\varepsilon\to 0} \mathrm{AG}_\varepsilon(u)$ pour $u \in \mathrm{BV}(\Omega, \mathbb{R}^2)$.

Semi-continuité des énergies de ligne La question de la semi-continuité des énergies de ligne pour une fonction coût générale est intéressante en soi mais aussi de par ses applications théoriques. Par exemple, une condition nécessaire pour une Γ-limite est la semi-continuité inférieure (voir Proposition 0.0.2). S'il est préssenti qu'une énergie de ligne E_c est la Γ-limite d'une énergie libre, telle que la fonctionnelle d'Aviles-Giga, c'est en particulier que cette dernière est semi-continue inférieurement (s.c.i.). Cette question, à priori plus simple que la question de la Γ-convergence d'une énergie libre est donc fondamentale. Il s'avère que très peu de résultats sont connus à ce jour et nous sommes très loin, semble-t-il, d'établir une condition nécessaire et suffisante générale pour la semi-continuité des énergies de ligne. La semi-continuité a été démontrée dans les seuls cas $c(t) = t^3$, étudié par P. Aviles et et Y. Giga [7], et $c(t) = t^2$, exploré récemment par R. Ignat et B. Merlet [37]. Dans [4], la semi-continuité avait été conjecturée dans le cas $c(t) = t^p$ avec $1 \leq p \leq 3$ et infirmée dans le cas $p > 3$. Dans le chapitre 6, nous verrons que cette propriété fait également défaut dans le cas $p < 1$.

Borne supérieure Concernant la borne supérieure pour la fonctionnelle d'Aviles-Giga, celle-ci demeure un problème ouvert. En particulier, il se pourrait que l'énergie se concentre, outre sur les ensembles de codimension 1, ici des lignes, sur des ensembles fractals de type Cantor, c'est à dire dont la codimension est comprise strictement entre 0 et 1.

Compacité La compacité pour la suite de fonctionnelles $(\mathrm{AG}_\varepsilon)_{\varepsilon>0}$ a par la suite été démontrée par L. Ambrosio, C. De Lellis et C. Mantegazza dans [4] et A. DeSimone, R. V. Kohn, S. Müller et F. Otto dans [28] à l'aide d'un principe de compacité par compensation et d'une notion d'entropie régulière sur $\mathbb{R}^2$.

Théorème. *La suite de fonctionnelles* $(\mathrm{AG}_\varepsilon)_{\varepsilon>0}$ *vérifie la propriété de compacité au sens de la Proposition 0.0.2 : toute suite d'énergie bornée, i.e. vérifiant* (0.0.5)*, est relativement compacte dans* $L^1(\Omega,\mathbb{R}^2)$*.*

Généralisation

Dans le chapitre 2, nous nous intéresserons en particulier à une généralisation du modèle d'Aviles-Giga de la forme

$$E_\varepsilon(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega \varepsilon|\nabla u(x)|^2 + \frac{1}{\varepsilon}W(u)\,\mathrm{d}x & \text{si } u \in H^1(\Omega,\mathbb{R}^2) \text{ et } \nabla\cdot u = 0,\\ +\infty & \text{sinon,} \end{cases}$$

où $\Omega \subset \mathbb{R}^2$ est un domaine borné et $W : \mathbb{R}^2 \to \mathbb{R}^+$ est le potentiel. Dans le modèle d'Aviles-Giga, c'est à dire pour le potentiel spécifique $W(u) = (1-|u|^2)^2$, il se trouve que le profil $1D$ est optimal pour la transition entre deux puits $u^\pm \in \mathbb{S}^1$. Cette propriété est fausse, cependant, pour un potentiel général. Une question qui nous intéressera dans la deuxième partie (en particulier dans le chapitre 7) concerne l'existence de certaines conditions sur le potentiel qui assurent que le profil $1D$ est optimal. Nous montrerons même, dans certains cas, qu'il est unique. Plus précisément, fixons deux puits $u^\pm \in \{z \in \mathbb{R}^2 : W(z) = 0\}$, disons $u^\pm = (a, b^\pm)$ pour simplifier : on peut se ramener à ce cas en se plaçant dans le repère $(\nu,\nu^\perp)$ où ν est le vecteur $u^+ + u^-$ normalisé. Sous certaines conditions fortes sur le potentiel W, nous montrerons que l'unique minimiseur global du problème suivant,

$$\inf\left\{\frac{1}{2}\int_\Omega |\nabla u|^2 + W(u)\ :\ u:\Omega\to\mathbb{R}^2,\ u(\pm\infty,\cdot) = u^\pm\right\},$$

ne dépend que de la première variable x_1, où $\Omega = \mathbb{R}\times\mathbb{R}/\mathbb{Z}$ est un cylindre infini dans la direction de x_1. Notons que si u ne dépend que d'une variable, disons $u(x) = u(x\cdot\nu)$ avec $\nu \in \mathbb{S}^1$, puisque $\nabla\cdot u = 0$, on a nécessairement $u\cdot\nu = cte$ et donc $\nu = \pm e_1$. Ce type de questions est généralement connu sous l'appélation "conjecture de De Giorgi" et concerne plus spécifiquement la symmétrie $1D$ pour les solutions scalaires de certaines équations elliptiques semi-linéaires sur l'espace $\mathbb{R}^d$ tout entier. Notre cadre est différent puisque nous considérons des champs de vecteurs à divergence nulle et périodiques en une variable. L'outil clef qui nous permettra d'aboutir à de tels résultats est la méthode d'entropie, qui remonte aux travaux de P. Aviles, Y. Giga, W. Jin et R. V. Kohn.

Micromagnétisme

Comme il a été introduit dans [6], le modèle d'Aviles-Giga est un modèle simplifié de cristaux liquides. Ce modèle peut aussi être vu comme un modèle jouet pour l'étude du ferromagnétisme telle que nous la présentons ici, dans les grandes lignes.

Certains matériaux, appelés *ferromagnétiques*, ont la capacité de s'aimanter sous l'effet d'un champ extérieur et de garder cette *aimentation* en mémoire. À l'échelle microscopique, chaque électron est caractérisé non seulement par sa charge mais aussi par son moment magnétique (qui provient du *spin*). Lorsque ces moments s'alignent dans une petite région, alors appelée *domaine magnétique*, ils créent un champ magnétique, l'aimentation, observable à l'échelle mésoscopique voir macroscopique. Afin de diminuer leur énergie interne, fonction en particulier du champ extérieur, les matériaux ferromagnétiques peuvent se séparer en plusieurs domaines magnétiques par des interfaces appelées *parois* du domaine. Pour une étude approfondie du ferromagnétisme et pour des résultats expérimentaux comportant de nombreux exemples de domaines magnétiques, le lecteur pourra se tourner vers le livre de A. Hubert et R. Schafer [35].

L'état d'un échantillon ferromagnétique, représenté par un domaine borné $\Omega \subset \mathbb{R}^3$, est caractérisé par une application $m : \Omega \to \mathbb{S}^2$, l'*aimentation* (ou *magnetisation* en anglais). En l'absence de champ extérieur, la théorie du micromagnétisme nous enseigne que m est un état stable de l'énergie libre, dite énergie de *Brown*, suivante

$$F_\varepsilon(m) = d^2 \int_\Omega |\nabla m|^2 + \int_\Omega \phi(m) + \int_{\mathbb{R}^3} |H|^2 \quad ,$$

où

- d est un paramètre dépendant de l'échantillon ferromagnétique appelé *longueur d'échange*. d est généralement négligeable devant la taille du domaine. Le premier terme de l'énergie micromagnétique, appelé *énergie d'échange*, pénalise les variations de m.
- $\phi : \mathbb{S}^2 \to \mathbb{R}^+$ est une application régulière, appelée *anisotropie*, qui tient compte de la structure cristalline de l'échantillon ferromagnétique en favorisant certaines directions, appelées *directions faciles d'aimentation*. Ces directions forment un ensemble, supposé non vide, de points annulant la fonction ϕ.
- $H \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ est un champ de vecteurs qui représente le champ magnétique induit par m sur l'espace $\mathbb{R}^3$ tout entier. Le dernier terme de l'énergie, $\int_{\mathbb{R}^3} |H|^2$, est appelé energie *magnétostatique*. H est déterminé par le système d'équations suivant :

$$\begin{cases} \nabla \times H = 0 & \text{sur } \mathbb{R}^3, \\ \nabla \cdot H = -\nabla \cdot (m\mathbf{1}_\Omega) & \text{sur } \mathbb{R}^3. \end{cases}$$

  Autrement dit, H est un champ gradient : $H = -\nabla u$, où $u : \mathbb{R}^3 \to \mathbb{R}$ est solution de l'équation $\Delta u = \nabla \cdot (m\mathbf{1}_\Omega)$ dans l'espace $\mathbb{R}^3$ tout entier. En particulier, $\int |H|^2 = \|\nabla \cdot (m\mathbf{1}_\Omega)\|^2_{H^{-1}(\Omega)}$ de telle sorte que l'énergie magnétostatique pénalise la divergence de $m\mathbf{1}_\Omega$ au sens des distributions.

Dans le cas particulier d'un échantillon ferromagnétique épais, assimilable à un cylindre infini $\Omega = \omega \times \mathbb{R}$ pour un domaine borné $\omega \subset \mathbb{R}^2$, où on fait de plus l'hypothèse que l'aimantation ne dépend pas de la variable d'épaisseur, on obtient un modèle $2D$ défini sur le domaine ω. Dans certains régimes où l'énergie magnétostatique est très forte, un modèle simplifié consiste alors à remplacer l'énergie magnétostatique par la contrainte $\nabla \cdot (\mathbf{1}_\omega m) = 0$ (en considérant ici que m est défini sur ω et pas Ω), i.e. $\nabla \cdot m = 0$ dans ω et $m \cdot n = 0$ au bord (condition de Neummann). Pour une anisotropie nulle sur le cercle $\mathbb{S}^1$, après normalisation des différents termes, on obtient une énergie de la forme suivante : pour tout $m = (m', m_3) \in L^1(\omega, \mathbb{S}^2)$,

$$E_\varepsilon(m) = \begin{cases} \dfrac{1}{2} \displaystyle\int_\omega \varepsilon |\nabla m|^2 + \frac{1}{\varepsilon} \varphi(m) & \text{si } m \in H^1(\omega, \mathbb{S}^2) \text{ et } \nabla \cdot m' = 0, \\ +\infty & \text{sinon,} \end{cases}$$

définie sur un domaine borné $\omega \subset \mathbb{R}^2$. Ici, l'anisotropie $\varphi : \mathbb{S}^2 \to \mathbb{R}^+$ vérifie

$$\begin{cases} \varphi(z) = 0 & \text{si } z \in \mathbb{S}^1 := \{\mathbb{S}^2 \ : \ z_3 = 0\}, \\ \varphi(z) > 0 & \text{sinon.} \end{cases}$$

À la limite lorsque $\varepsilon \to 0$, on s'attend à rencontrer des champs de vecteurs unitaires $m : \omega \to \mathbb{S}^1$ à divergence nulle, exactement comme dans le modèle d'Aviles-Giga. Dans le cas le plus simple, comme dans le modèle d'Aviles-Giga, le profil $1D$ est optimal. Notons qu'étant donné la contrainte sur la divergence, dès que l'aimentation est $1D$, disons $m(x) = m(x_1)$, la première composante de m est constante si bien que m tourne dans un plan orthogonal au premier axe de coordonnées (le plan de la paroi), entre les deux valeurs de l'aimantation (entre le premier et le deuxième domaine). De telles transitions au sein des couches limites (interface d'épaisseur $\sim \varepsilon$ entre deux domaines magnétiques) sont appelées *parois de Bloch* (voir figure 4). En général, cependant, la microstructure formée par un minimiseur de l'énergie E_ε au sein d'une couche limite pourrait ne pas être $1D$. En effet de nombreux tests expérimentaux révèlent la présence de structure $2D$ plus ou moins complexes (voir [35]). Certaines de ces structures $2D$ ont également été étudiées théoriquement comme les parois de type "cross-tie" [2, 57] ou encore les parois en motifs de "zigzag" (voir [38]) par exemple.

Pour une anisotropie de la forme $\varphi(m) = |m_3|^\alpha$ avec $0 < \alpha \leq 4$, certains travaux ([36, 4]) laissent à penser que les parois de Bloch sont optimales. Le cas particulier $\varphi(m) = |m_3|^2$ a été étudié par R. Ignat and B. Merlet dans [36] où l'optimalité des parois de Bloch a été démontrée dans le cas de la configuration de "saut maximal", c'est à dire pour la transition entre $-\nu$ et ν avec $\nu \in \mathbb{S}^1$. Pour finir, remarquons que dans le cas $\varphi(m) = m_3^4 = (1 - |m'|^2)^2$ (où $m' = (m_1, m_2)$), E_ε n'est autre que la fonctionnelle d'Aviles-Giga à laquelle un terme a été ajouté puisque

$$E_\varepsilon(m) = \mathrm{AG}_\varepsilon(m') + \frac{\varepsilon}{2} \int_\omega |\nabla m_3|^2 = \mathrm{AG}_\varepsilon(m') + \frac{\varepsilon}{2} \int_\omega \left| \nabla \left(\sqrt{1 - |m'|^2} \right) \right|^2 \quad .$$

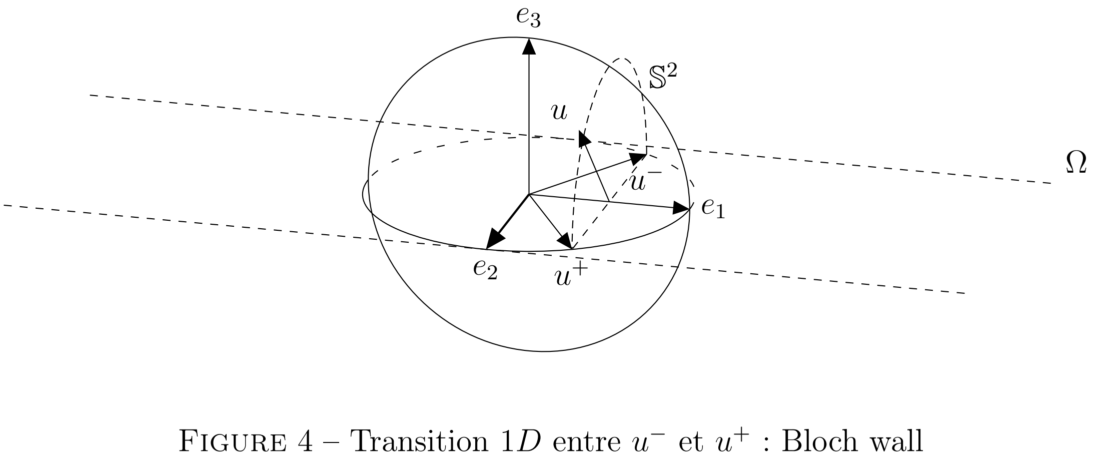

FIGURE 4 – Transition $1D$ entre u^- et u^+ : Bloch wall

Une approximation du transport branché

Transport branché Le problème du transport branché consiste à déterminer une méthode optimale pour déplacer une distribution de masse donnée vers une autre en supposant que le coût pour le transport d'une masse m par unité de longueur est proportionnel à m^α pour $0 \leq \alpha < 1$ et non à m comme c'est le cas pour des modèles de transport plus standards comme dans la théorie de Monge-Kantorovich. La sous-additivité de la fonction coût, $m \mapsto m^\alpha$, force les masses à se regrouper puis à se déplacer ensemble aussi longtemps que possible jusqu'à ce qu'elles se séparent à nouveau vers leur différentes destinations justifiant ainsi l'expression "transport branché". Ce type de problématique apparaît dans différentes situations : dans la nature (bassins hydrographiques, vaisseaux sanguins...) et dans certaines structures construites par l'homme où l'économie d'échelle rend la construction des grandes routes proportionnellement moins chère que celle des petites routes.

Cette théorie a d'abord été étudiée dans le cadre discret par Gilbert dans [34]. Dans ce cadre, les distributions “source” et “cible” sont représentées par deux collections de points x_i et y_j dans $\Omega \subset \mathbb{R}^d$ auxquels sont associés des masses a_i et b_j : chaque x_i émet une masse a_i et chaque y_j absorbe une masse b_j. Un réseau de transport de la distribution source vers la distribution cible est alors représenté par un graphe pondéré et orienté $G = (E(G), \theta)$ ($E(G)$ est un ensemble d'arêtes et $\theta : E(G) \to \mathbb{R}^+$ est une application qui leur associe un poids) qui satisfait les lois de Kirchhoff : en chaque noeud la masse qui arrive est égale à la masse qui repart, modulo la masse a_i, émise en chaque sommet x_i, et la masse b_j absorbée en chaque sommet y_j. Parmis tous ces graphes, on cherche à minimiser l'énergie totale $E^\alpha(G) := \sum_{e \in E(G)} \theta(e)^\alpha |e|$, où $|e|$ est la longueur de e. Le cas particulier $\alpha = 0$ correspond au célèbre problème de Steiner qui consiste à déterminer un réseaux de longueur minimale reliant un ensemble fini de points donnés.

Lorsque le nombre de points est très élevé, la complexité des algorithmes issus de l'optimisation combinatoire les rend inutilisables pour trouver une solution au problème en temps raisonnable. C'est pourquoi nous préférons une approche de type calcul des variation qui s'appuie sur une formulation continue du transport branché : les distributions de masse sont représentées par des densités. La première approche variationnelle est due à Q. Xia dans [64] où il propose un modèle eulérien défini sur les mesures vectorielles. Presque au même moment, F. Maddalena, J.M. Morel et S. Solimini donnèrent une formulation lagrangienne utilisant des mesures sur les chemins dans [47]. Plusieurs autres approches ou généralisations ont ensuite été introduites [10, 11, 12] et diverses applications ont été découvertes [58, 13]. Dans cette thèse, nous considèrerons seulement le modèle eulérien proposé par Q. Xia. Ceci n'est cependant pas restrictif puisque l'équivalence de tous ces modèles, eulériens ou lagrangiens a par la suite été démontrée [12, 54].

L'idée principale de Q. Xia dans son modèle eulerien est de traduire la contrainte, donnée par les lois de Kirchhoff dans le modèle de Gilbert, par une condition sur la divergence du flot de masse. Plus précisément, étant donné deux mesures de probabilité μ et ν sur $\Omega \subset \mathbb{R}^d$, un réseau de transport continu entre μ et ν est représenté par une mesure vectorielle u sur Ω telle que $\nabla \cdot u = \mu - \nu$. L'énergie du transport branché est alors donnée par

$$M^\alpha(u) = \begin{cases} \displaystyle\int_M \theta^\alpha \,\mathrm{d}\mathcal{H}^1 & \text{if } u = U(M, \theta, \xi), \\ +\infty & \text{sinon,} \end{cases}$$

où $U(M, \theta, \xi)$ est la mesure vectorielle rectifiable $\theta\xi \cdot \mathcal{H}^1_{|M}$ avec densité θ par rapport à la mesure de Hausdorff $\mathcal{H}^1$ sur l'ensemble rectifiable M, $\theta : M \to \mathbb{R}^+$ est la multiplicité et $\xi : M \to \mathbb{S}^{d-1}$ est l'orientation. Notons que l'irrigation d'une densité à partir d'une autre (c'est à dire la détermination d'une mesure optimale pour le problème précédent) est d'autant plus difficile que α est proche de 0 puisque le coût θ^α (avec $\theta \in [0,1]$) croît lorsque α décroît. Dans [64], Q. Xia a en particulier démontré que toute mesure de probabilité μ peut être irriguée à partir d'une masse de Dirac dès lors que α est suffisamment proche de 1, à savoir $\alpha > 1 - \frac{1}{d}$.

Approximation Récemment, E. Oudet et F. Santambrogio ont réussi à mettre au point une méthode numérique efficace pour approcher le problème précédent (minimisation de la fonctionnelle singulière M^α). Il ont pour cela introduit une suite d'approximations M_ε^α définies de la manière suivante,

$$M_\varepsilon^\alpha(u) = \varepsilon^{-\gamma_1} \int_\Omega |u|^\beta + \varepsilon^{\gamma_2} \int_\Omega |\nabla u|^2,$$

où $\beta \in (0,1)$ et $\gamma_1, \gamma_2 > 0$ sont des exposants dépendant de la dimension d et de α. Cette fois, l'énergie approximée M_ε^α est définie pour des champs de vecteurs réguliers avec contrainte de divergence plutôt que pour des mesures singulières. La Γ-convergence de ces énergies vers $c_0 M^\alpha$, où $c_0 > 0$ est une constante, a été démontrée dans [53] en dimension 2. Ces résultats ont permis à E. Oudet de mettre en place une méthode numérique efficace basée sur la minimisation des fonctionnelles M_ε^α : à partir d'une valeur de ε préalablement fixée, assez grande pour que le terme convexe $\varepsilon \int |\nabla u|^2$ soit dominant, un minimum de M_ε^α est approché par une méthode de descente (de gradient) ; la valeur de ε est ensuite réduite progressivement, en suivant une méthode de continuation, et la descente du gradient est initialisée avec le minimum obtenu à l'étape précédente. En revanche, bien que des simulations numériques très satisfaisantes aient été obtenues (les figures 5 et 6 en illustrent quelques-unes), ces résultats ne prenaient pas en compte la contrainte de divergence. Notons qu'ici, la contrainte $\nabla \cdot u = \mu - \nu$ doit être remplacée par $\nabla \cdot u = f_\varepsilon - g_\varepsilon$ pour certaines approximations f_ε (resp. g_ε) de μ (resp. ν) dans L^2 puisque la divergence d'un champ de vecteurs $u \in H^1$ est toujours dans L^2. Dans la première partie, après avoir étendu le résultat de Γ-convergence en toute dimension, nous démontrerons, sous certaines hypothèses sur f_ε et g_ε, que la Γ-convergence a encore lieu pour la fonctionnelle $\overline{M}_\varepsilon^\alpha$ définie avec contrainte de divergence $\nabla \cdot u = f_\varepsilon - g_\varepsilon$ ($\overline{M}_\varepsilon^\alpha(u) = +\infty$ dès que $\nabla \cdot u \neq f_\varepsilon - g_\varepsilon$). En effet, bien que l'ajout de la contrainte de volume dans le résultat de Γ-convergence de Modica-Mortola n'est pas très difficile, nous verrons que la situation est bien plus délicate pour les fonctionnelles M_ε^α et la contrainte $\nabla \cdot u = f_\varepsilon - g_\varepsilon$.

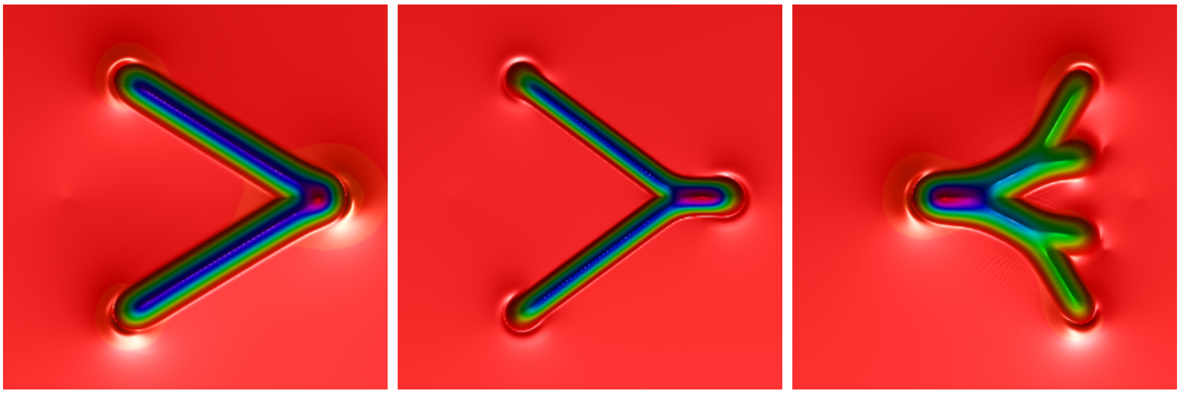

Figure 5 – Simulations obtenues par la méthode de Γ-convergence décrite dans cette introduction : irrigation de deux puis quatre sources ponctuelles à partir d'une seule (issu de l'article [53] de la revue *Arch. Ration. Mech. Anal.*)

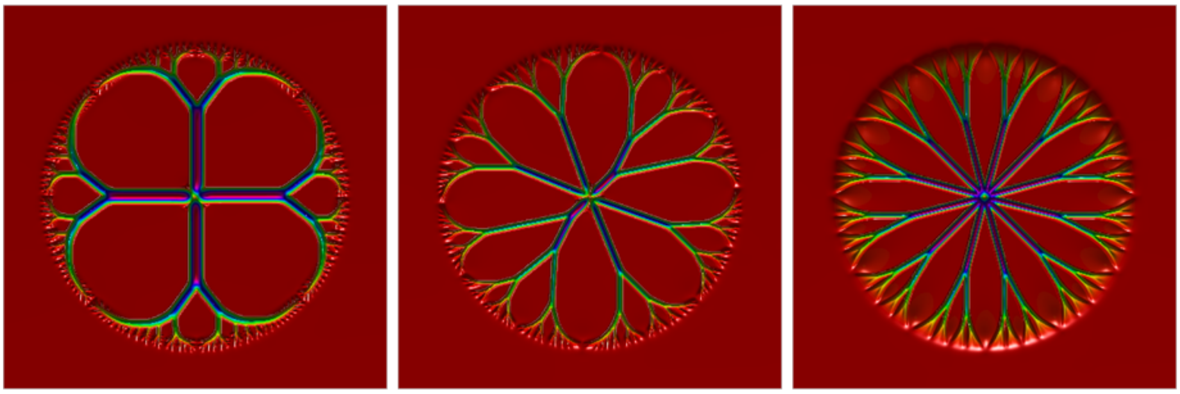

Figure 6 – Simulations obtenues par la méthode de Γ-convergence décrite dans cette introduction : irrigation de la mesure uniforme sur le cercle à partir d'une source en son centre pour différentes valeurs de l'exposant α (issu de l'article [53] de la revue *Arch. Ration. Mech. Anal.*)

Part I

A phase-field approximation of branched transportation

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UNE APPROXIMATION DE TYPE MODICA-MORTOLA DU TRANSPORT BRANCHÉ

Résumé

Les modèles de transport branché sont souvent exprimés comme un problème de minimisation d'une énergie M^α définie sur les mesures vectorielles concentrées sur un ensemble 1-rectifiable avec une contrainte de divergence. Nous étudions des approximations de type Modica-Mortola des énergies M^α introduites par Edouard Oudet et Filippo Santambrogio dans le cas de la dimension 2. Ces énergies, notées M_ε^α, sont définies pour des champs de vecteurs dans H^1. Dans cette partie, nous étendons leur résultat de Γ-convergence en toute de dimension. Par ailleurs, nous introduisons une famille de pseudo-distances définies sur les densités de probabilité L^2 à travers le problème de minimisation $\min\{M_\varepsilon^\alpha(u) \ : \ \nabla\cdot u = f^+ - f^-\}$. Nous prouvons des estimations uniformes sur ces pseudo-distances qui permettent d'établir un résultat de Γ-convergence pour M_ε^α avec contrainte de divergence.

Abstract

Models for branched networks are often expressed as the minimization of an energy M^α over vector measures concentrated on 1-dimensional rectifiable sets with a divergence constraint. We study some Modica-Mortola type approximations of M^α introduced in the two dimensional case by Edouard Oudet and Filippo Santambrogio. These energies, denoted by M_ε^α, are defined over H^1 vector measures. In this part, we extend their Γ-convergence result to every dimension. We also introduce some pseudo-distances between L^2 densities obtained through the minimization problem $\min\{M_\varepsilon^\alpha(u) \ : \ \nabla\cdot u = f^+ - f^-\}$. We prove some uniform estimates on these pseudo-distances which allow us to establish a Γ-convergence result for M_ε^α with a divergence constraint.

Structure of this part In a short introduction, we recall Xia's formulation of branched transportation and its approximations M_ε^α introduced by E. Oudet and F. Santambrogio. In chapter 2, we extend the Γ-convergence result of E. Oudet and F. Santambrogio in every dimension. The longest part of this chapter, section 3.2, is devoted to a local estimate which gives a bound on the minimum value $d_\varepsilon^\alpha(f^+, f^-) := \min\{M_\varepsilon^\alpha(u) \ : \ \nabla\cdot u = f^+ - f^-\}$ depending on $\|f\|_{L^1}$, $\|f\|_{L^2}$ and $\mathrm{diam}(\Omega)$ (see Proposition 3.2.2 page 60). In section 3.3, we deduce a comparison between d_ε^α and the Wasserstein distance with an "error term" involving the L^2 norm of $f^+ - f^-$. As an application of this inequality, in the last chapter of this part, we will prove the Γ-convergence result which was lacking in [53], of functionals $\overline{M}_\varepsilon^\alpha$ to $\overline{M}^\alpha$ (with a divergence constraint on $\nabla\cdot u$): this answers the Open question 1 in [59, 53] and validates their numerical method.

Chapter 1

Introduction

In this chapter, we are interested in some approximation of branched transportation proposed by E. Oudet and F. Santambrogio few years ago in [53] and which has interesting numerical applications. This model was inspired by the well known scalar phase transition model proposed by L. Modica and S. Mortola in [48]. Given $u \in H^1(\Omega, \mathbb{R}^d)$ for some bounded open subset $\Omega \subset \mathbb{R}^d$, E. Oudet and F. Santambrogio introduced the following energy:

$$M_\varepsilon^\alpha(u) = \varepsilon^{-\gamma_1} \int_\Omega |u|^\beta + \varepsilon^{\gamma_2} \int_\Omega |\nabla u|^2,$$

where $\beta \in (0,1)$ and $\gamma_1, \gamma_2 > 0$ are some exponents depending on α (see (1.2.2)). It was proved in [53] that, at least in two dimensions, the energy sequence $(M_\varepsilon^\alpha)_{\varepsilon>0}$ Γ-converges to the branched transportation functional $c_0 M^\alpha$ for some constant c_0 and for some suitable topology (see Theorem 1.2.1 page 42). This result has been interestingly applied to produce a numerical method. However, rather than a Γ-convergence result on M_ε^α we would need to deal with the functionals $\overline{M}_\varepsilon^\alpha$, obtained by adding a divergence constraint: it should be shown that $\overline{M}_\varepsilon^\alpha(u) := M_\varepsilon^\alpha(u) + I_{\nabla \cdot u = f_\varepsilon}$ Γ-converges to $\overline{M}^\alpha(u) := M_\varepsilon^\alpha(u) + I_{\nabla \cdot u = \mu^+ - \mu^-}$, where $\mu^\pm$ are two probability measures, $f_\varepsilon \in L^2$ is some suitable approximation of $\mu^+ - \mu^-$ and $I_A(u)$ is the indicator function in the sense of convex analysis that is 0 whenever the condition is satisfied and $+\infty$ otherwise. Even if this property was not proved in [53], the effectiveness of the numerical simulations made the authors think that it actually holds true. Note that an alternative using a penalization term was proposed in [59] to overcome this difficulty. Recently, many other phase-field type models for optimal networks, based on the same kind of considerations, have been proposed (see [45] for an approximation of the Steiner problem, [20] for an approximation of the Willmore functional).

We are going to remind a few properties and tools in the branched transportation theory (see [12] for further study and full demonstrations of the claimed properties). Then, we will state our main results concerning the functional M_ε^α and M^α with and without divergence constraints.

1.1 Branched transportation theory: an overlook

Branched transportation is a classical problem in optimization: it is a variant of the Monge-Kantorovich optimal transportation theory in which the transport cost for a mass m per unit of length is not linear anymore but sub-additive. More precisely, the cost to transport a mass m on a length l is considered to be proportional to $m^\alpha l$ for some $\alpha \in]0,1[$. As a result, it is more efficient to transport two masses m_1 and m_2 together instead of transporting them separately. For this reason, an optimal pattern for this problem has a "graph structure" with branching points. Contrary to what happens in the Monge-Kantorovich model, in the setting of branched transportation, an optimal structure cannot be described only using a transport plan, giving the correspondence between origins and destinations, but we need a model which encodes all the trajectories of mass particles.

Branched transportation theory is motivated by many structures that can be found in the nature: vessels, trees, river basins… Similarly, as a consequence of the economy of scale, large roads are proportionally cheaper than large ones and it follows that the road and train network also present this structure. Surprisingly the theory has also had theoretical applications: recently, it has been used by F. Bethuel in [13] so as to study the density of smooth maps in Sobolev spaces between manifolds.

Branched transportation theory was first introduced in the discrete framework by E. N. Gilbert in [34] as a generalization of the Steiner problem. In this case an admissible structure is a weighted graph composed of oriented edges of length l_i on which some mass m_i is flowing. The cost associated to it is then $\sum_i l_i m_i^\alpha$ and it has to be minimized over all graphs which transport some given atomic measure to another one. More recently, the branched transportation problem was generalized to the continuous framework by Q. Xia in [64] by means of a relaxation of the discrete energy (see also [65]). Then, many other models and generalizations have been introduced (see [47] for a Lagrangian formulation, see also [10], [11], [12] for different generalizations and regularity properties). In this chapter, we will concentrate on the model with a divergence constraint, due to Q. Xia. However, this is not restrictive since all these models have been proved to be equivalent (see [12], [54]). Concerning the problem of switching from the Eulerian to the Lagrangian model we also point out a recent work of F. Santambrogio [60] who gives a new proof of the Smirnov decomposition Theorem.

An interesting model related to the branched transportation theory is the ramified allocation problem which aims at finding an optimal allocation plan for transporting commodity from factories to households: Given a finite set of points X representing factories and a probability measure μ representing households, we look for an allocation plan q which minimizes the branched transportation energy among all transport paths compatible with q and connecting a probability measure concentrated on X to the measure μ (see [67]). A significant difference with the branched transportation problem, presented above, is that the production of each factories is not prescribed: only their locations (points in the set X) are known.

1.1.1 The discrete model (Gilbert)

Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$ being the dimension, be a bounded open set and let us fix two atomic probability measures $\mu^\pm$ on Ω:

$$\mu^+ = \sum_{i=1}^{I^+} m_i^+ \delta_{x_i^+} \quad \text{and} \quad \mu^- = \sum_{i=1}^{I^-} m_i^- \delta_{x_i^-} \quad ,$$

where $I^\pm \in \mathbb{N}^*$, x_i^+ (resp. x_i^-) are distinct points in Ω and the masses $m_i^\pm \in (0,1]$ satisfy $\sum_{i=1}^{I^\pm} m_i^\pm = 1$. We want to connect μ^- to μ^+ by a weighted oriented graph $G = (E(G),\theta)$: $E(G)$ is a finite set of oriented edges $\mathbf{e} = (a_\mathbf{e}, b_\mathbf{e})$ for some points $a_\mathbf{e}, b_\mathbf{e} \in \Omega$ and $\theta : E(G) \to (0,+\infty)$ is the weight function. $a_\mathbf{e}$ (resp. $b_\mathbf{e}$) is called starting (resp. finishing) endpoint of the edge $\mathbf{e}$. Any point in Ω which is the endpoint of at least one edge $\mathbf{e} \in E(G)$ is called *vertex* of G. Given a weighted oriented graph G, the set of all its vertices is denoted by $V(G)$. The support of $\mathbf{e}$ (in bold) is denoted by $e = [a_\mathbf{e}, b_\mathbf{e}]$ and called edge. Last of all, the orientation of $\mathbf{e}$ is denoted by $\tau_\mathbf{e} := \frac{b_\mathbf{e} - a_\mathbf{e}}{|b_\mathbf{e} - a_\mathbf{e}|}$.

Definition 1.1.1. We say that a weighted oriented graph $G = (E(G),\theta)$ *irrigates* μ^+ from μ^- or is a *transport path* from μ^- to μ^+ if for all point $v \in \Omega$, G satisfies the Kirchhoff laws:

$$\sum_{\mathbf{e}=(a_\mathbf{e},v)\in E(G)} \theta(\mathbf{e}) - \sum_{\mathbf{e}=(v,b_\mathbf{e})\in E(G)} \theta(\mathbf{e}) = \begin{cases} m_i^+ & \text{if } v = x_i^+ \text{ and } v \notin \operatorname{supp}(\mu^-), \\ -m_i^- & \text{if } v = y_j^- \text{ and } v \notin \operatorname{supp}(\mu^+), \\ m_i^+ - m_i^- & \text{if } v = x_i^+ = x_j^-, \\ 0 & \text{otherwise.} \end{cases}$$

The set of all weighted oriented graphs G irrigating μ^+ from μ^- is denoted by $\mathcal{G}(\mu^-, \mu^+)$.

The first term in the preceding equation (Kirchhoff laws) represents the difference between incoming and outcoming mass at v while the second term is the mass of the measure $\mu^+ - \mu^-$ at v. Note that both terms vanish when v is not a vertex of G and does belong to the supports of μ^+ and μ^-. Let us fix $\alpha \in [0,1]$. The energy of $G = (E(G),\theta)$ is defined by

$$\mathcal{E}^\alpha(G) := \sum_{\mathbf{e}=(a_\mathbf{e},b_\mathbf{e})\in E(G)} \theta(\mathbf{e})^\alpha \operatorname{length}(e).$$

Our goal is to minimize $\mathcal{E}^\alpha$ among all weighted oriented graphs G irrigating μ^+ from μ^-:

$$\min\{\mathcal{E}^\alpha(G) \ : \ G \in \mathcal{G}(\mu^-, \mu^+)\}. \tag{1.1.1}$$

When $\alpha = 0$ we find the well-known Steiner's problem which corresponds to minimizing the total length of the graph G. Our model is a generalization of Steiner's problem where the cost also depends on the mass flowing on each edge. More precisely, the cost for moving a mass m on length l is equal to $m^\alpha l$, $\alpha \in [0,1]$. In order to get the existence of a minimizer for (1.1.1), we need to avoid cycles in G. Indeed, since we work with finite dimensional objects, in order to get compactness we have to get a uniform bound on the dimension, i.e. on the number of vertices of G. To this aim, we need the following definitions:

Definition 1.1.2. Let $G = (E(G), \theta)$ be a weighted oriented graph.

sub-graph: A sub-graph of G is weighted oriented graph $G' = (E(G'), \theta')$ such that $E(G') \subset E(G)$ and $\theta'(\mathbf{e}) = \theta(\mathbf{e})$ for all $\mathbf{e} \in E(G')$.

cycle: A cycle G' of G is a non trivial sub-graph of G composed of a sequence of adjacent edges. More precisely, we impose that the support of G' (i.e. the union of all edges in $E(G')$) is connected and each vertex of G' has multiplicity 2: it is the endpoint of exactly two edges of G'.

circuit: A circuit is a sequence of oriented edges which are adjacent (taking into account the orientation). More precisely, G' is a cycle and each vertex is the starting endpoint and the finishing end point of two distinct edges.

It is not difficult to see, using the fact that $m \to m^\alpha$ is non-decreasing, that the energy of G can be reduced by removing circuits. Note that this property requires that $\alpha \geq 0$. The case $\alpha < 0$, which is not considered in this thesis, is also interesting and may have applications. If $\alpha < 0$, contrary to what happens when $\alpha \geq 0$, optimal paths may prefer to have circuits (see [66]).

In our situation, using the concavity of the cost function $m \to m^\alpha$, one can even remove all cycles in G:

Lemma 1.1.3. *Let $G \in \mathcal{G}(\mu^-, \mu^+)$ be a weighted oriented graph irrigating μ^+ from μ^-. There exists a sub-graph $G' \in \mathcal{G}(\mu^-, \mu^+)$ of G such that $\mathcal{E}^\alpha(G') \leq \mathcal{E}^\alpha(G)$ and G' has no cycles.*

This easily provides the existence of a minimizer for the problem (1.1.1):

Proposition 1.1.4. *Let $\mu^\pm$ be two atomic probability measures. Then there exists a weighted oriented graph $G \in \mathcal{G}(\mu^-, \mu^+)$ such that*

$$\mathcal{E}^\alpha(G) = \min\{\mathcal{E}^\alpha(G') \ : \ G' \in \mathcal{G}(\mu^-, \mu^+)\}.$$

Moreover, G has no circuits and no cycles if $\alpha \in (0, 1)$.

A fundamental necessary condition satisfied by any optimal path $G \in \mathcal{G}(\mu^-, \mu^+)$ is the following: for any bifurcation point, i.e. $v \in V(G)$ which is neither in the support of μ^+ nor in that of μ^-, one has

$$\sum_{\mathbf{e}=(v,b_\mathbf{e})} \theta(\mathbf{e})^\alpha \tau_\mathbf{e} - \sum_{\mathbf{e}=(a_\mathbf{e},v)} \theta(\mathbf{e})^\alpha \tau_\mathbf{e} = 0.$$

This condition in particular allows to compute the angles between edges adjacent to some bifurcation point v. When $\alpha = 0$, i.e. for Steiner's problem, we get the classic condition that the angle between two consecutive edges adjacent to a bifurcation point is equal to $\pi/3$. By contrast, when $\alpha > 0$, these angles may depend on the incoming/outcoming mass of each adjacent edge. These conditions are useful, for instance in order to get a uniform bound on the number of edges adjacent to a bifurcation point (see section 12.3 in [12]).

1.1.2 The continuous model (Xia)

We briefly introduce the Xia model [64] obtained by relaxation of the discrete energy. We first give an eulerian model by transposing the discrete energy, defined over the space of oriented weighted graphs, to the space of vectorial measures. The main idea is that the Kirchhoff laws translate into a divergence constraint.

Let $G = (E(G), \theta)$ be a weighted oriented graph. We define the vector measure associated to G by

$$u_G := \sum_{\mathbf{e}\in E(g)} \theta(\mathbf{e})\tau_{\mathbf{e}}\, \mathrm{d}\mathcal{H}^1_{\llcorner e} \quad .$$

These measures u_G are called "transport paths" (see Definition 2.1 in [64]). u_G is characterized by its action on the space $\mathcal{C}(\overline{\Omega}, \mathbb{R}^d)$ by duality: for all $\varphi \in \mathcal{C}(\overline{\Omega}, \mathbb{R}^d)$

$$\langle u_G\, ; \varphi\rangle = \sum_{\mathbf{e}\in E(G)} \theta(\mathbf{e}) \int_e \varphi(x)\cdot\tau_{\mathbf{e}}\, \mathrm{d}\mathcal{H}^1(x) \quad .$$

Given $\mu^\pm$ two atomic probability measures, we remark that

$$G \in \mathcal{G}(\mu^-, \mu^+) \iff \nabla\cdot u_G = \mu^+ - \mu^- \quad .$$

For vector measures u which are *concentrated on a graph*, i.e. $u = u_G$ for some weighted oriented graph G, the *α-irrigation* energy of u (or branched transportation energy with exponent α) is defined by $M^\alpha(u_G) := \mathcal{E}^\alpha(G)$. We would want to extend this definition for any vector measure u. This was done in [64] by mean of a relaxation method. We first introduce some definitions:

Let $d \geq 1$ be the dimension and Ω be some open and bounded subset of $\mathbb{R}^d$. Let us denote by $\mathcal{M}_{div}(\Omega)$ the set of finite vector measures on $\overline{\Omega}$ such that their divergence is also a finite measure:

$$\mathcal{M}_{div}(\Omega) := \left\{u \text{ measure on } \overline{\Omega} \text{ valued in } \mathbb{R}^d \ :\ \|u\|_{\mathcal{M}_{div}(\Omega)} < +\infty\right\},$$

where $\|u\|_{\mathcal{M}_{div}(\Omega)} := |u|(\overline{\Omega}) + |\nabla\cdot u|(\overline{\Omega})$ with

$$|u|(\overline{\Omega}) := \sup\left\{\int_{\overline{\Omega}} \psi\cdot \mathrm{d}u \ :\ \psi \in \mathcal{C}(\overline{\Omega},\, \mathbb{R}^d),\ \|\psi\|_\infty \leq 1\right\}$$

and, similarly,

$$|\nabla\cdot u|(\overline{\Omega}) := \sup\left\{\int_{\overline{\Omega}} \nabla\varphi\cdot \mathrm{d}u \ :\ \varphi \in \mathcal{C}^1(\overline{\Omega}, \mathbb{R}),\ \|\varphi\|_\infty \leq 1\right\}.$$

In all what follows, $\nabla\cdot u$ has to be thought in the weak sense, i.e. $\int \varphi\nabla\cdot u = -\int \nabla\varphi\cdot \mathrm{d}u$ for all $\varphi \in \mathcal{C}^1(\overline{\Omega})$. Since we do not ask φ to vanish at the boundary, $\nabla\cdot u$ may contain possible parts on $\partial\Omega$ which are equal to $u\cdot n$ when u is smooth, where n is the external unit normal vector to $\partial\Omega$. In other words, $\nabla\cdot u$ is the weak divergence of $u\mathbf{1}_\Omega$ in $\mathbb{R}^d$, where $\mathbf{1}_\Omega$ is the indicator function of Ω, equal to 1 on Ω and 0 elsewhere. $\mathcal{M}_{div}(\Omega)$ is endowed with the topology of weak convergence on u and on its divergence: i.e. $u_n \overset{\mathcal{M}_{div}(\Omega)}{\longrightarrow} u$ if $u_n \rightharpoonup u$ and $\nabla\cdot u_n \rightharpoonup \nabla\cdot u$ weakly as measures.

We are know able to extend the definition of M^α to the whole space $\mathcal{M}_{div}(\Omega)$: for all $u \in \mathcal{M}_{div}(\Omega)$,

$$M^\alpha(u) := \inf \Big\{ \liminf_{n\to\infty} \mathcal{E}^\alpha(G_n) \ : \ G_n \text{ weighted oriented graphs s.t. } u_{G_n} \xrightarrow[n\to\infty]{} u \text{ in } \mathcal{M}_{div}(\Omega) \Big\}. \tag{1.1.2}$$

Given two probability measures $\mu^\pm$ on $\overline{\Omega}$, the branched transportation minimization problem becomes

$$\min \left\{ M^\alpha(u) \ : \ u \in \mathcal{M}_{div}(\Omega) \text{ s.t. } \nabla \cdot u = \mu^+ - \mu^- \right\}. \tag{1.1.3}$$

In this framework, the vector measure u with prescribe divergence must be considered as the momentum (the mass θ times the velocity) of a particle at some point. Then, $(\nabla \cdot u)(x)$ represents the difference between incoming and outcoming mass at each point x. Note that, if $\mu^\pm(\partial\Omega) = 0$, the divergence constraint implies a Neumann condition on u: $u \cdot n = 0$ on $\partial\Omega$.

Thanks to a classical rectifiability theorem by B. White, [63], one can prove that $M^\alpha(u) < \infty$ implies that u is $\mathcal{H}^1$-rectifiable. Moreover, one has the following representation formula for M^α (see also [65]):

Proposition 1.1.5. *For any vector measure $u \in \mathcal{M}_{div}(\Omega)$,*

$$M^\alpha(u) = \begin{cases} \int_M \theta(x)^\alpha \, \mathrm{d}\mathcal{H}^1(x) & \text{if } u \text{ can be written as } u = U(M, \theta, \xi), \\ +\infty & \text{otherwise,} \end{cases} \tag{1.1.4}$$

where $U(M, \theta, \xi)$ is the rectifiable vector measure $u = \theta\xi \cdot \mathcal{H}^1 \llcorner M$ with density $\theta\xi$ with respect to $\mathcal{H}^1$ on the rectifiable set M. The real multiplicity is a measurable function $\theta : M \to \mathbb{R}^+$ and the orientation $\xi : M \to S^{d-1} \subset \mathbb{R}^d$ is such that $\xi(x)$ is tangential to M for $\mathcal{H}^1$-a.e. $x \in M$. Note that the last tangential condition is a necessary condition for $\nabla \cdot u$ to be a measure.

In all the sequel we will use the definition (1.1.4) for M^α. Since M^α was initially expressed by relaxation in (1.1.2), we get freely the following density result:

Proposition 1.1.6. *The class of transport paths is dense in energy for M^α, that is: for all $u \in \mathcal{M}_{div}(\Omega)$, there exists a sequence $(v_n)_{n\geq 1} = (u_{G_n})_{n\geq 1} \subset \mathcal{M}_{div}(\Omega)$ of measures concentrated on weighted oriented graphs G_n such that*

$$v_n \xrightarrow[n\to\infty]{} u \quad \text{and} \quad M^\alpha(v_n) \xrightarrow[n\to\infty]{} M^\alpha(u) \quad .$$

This result is going to be useful in order to prove the Γ-convergence of the relaxed energies, M^α_ε, to M^α. In particular, a classical property in the theory of Γ-convergence states that, in order to get the upper bound, it is enough to find a recovery sequence for u belonging to a class of measures which are dense in energy.

1.1.3 Irrigability and irrigation distances

For the minimum value in (1.1.3) to be finite whatever μ^+ and μ^- in the set of probability measures, we will require α to be sufficiently close to 1.

Proposition 1.1.7 (Irrigability). *Assume that α satisfies the following inequalities,*

$$1 - \frac{1}{d} < \alpha < 1.$$

Then, for all probability measures μ^+, μ^- on $\overline{\Omega}$, there exists at least one vector measure $u \in \mathcal{M}_{div}(\Omega)$ such that $\nabla \cdot u = \mu^+ - \mu^-$ and $M^\alpha(u) < +\infty$.

Proof. We refer to [64] or Corollary 6.9. in [12] for a proof. □

In [64], Q. Xia has remarked that, as in optimal transportation theory, M^α induces a distance d^α on the space $\mathcal{P}(\overline{\Omega})$ of probability measures on $\overline{\Omega}$:

Definition 1.1.8. Given $\alpha \in (1 - \frac{1}{d}, 1)$, the *$\alpha$-irrigation distance* is defined by

$$d^\alpha(\mu^+, \mu^-) = \inf \left\{ M^\alpha(u) \ : \ u \in \mathcal{M}_{div}(\Omega) \quad \text{such that} \quad \nabla \cdot u = \mu^+ - \mu^- \right\}$$

for all probability measures $\mu^+, \mu^- \in \mathcal{P}(\overline{\Omega})$.

Thanks to our assumption $\alpha > 1 - 1/d$, d^α is finite for all $\mu^\pm \in \mathcal{P}(\overline{\Omega})$. The fact that d^α is a distance is quite easy considering that $m \to m^\alpha$ is subadditive. More precisely, we have the following result

Proposition 1.1.9. *d^α is a distance on the set $\mathcal{P}(\overline{\Omega})$ which metrizes the topology of weak convergence of measures.*

1.1.4 Monge-Kantorovich problem, comparison between irrigation and Wasserstein distances

We shall give a brief overlook of the Monge and Kantorovich problems which were introduced before the branched transportation theory. In [49], G. Monge addressed the question of finding an optimal way for moving a pile of sand from some place to another one with a new shape. His only axiom were that the cost for moving a mass is proportional to the mass time the distance covered (which corresponds to $\alpha = 1$ in our model). Such a transport scheme can be described by a map which allocates a destination to each initial point. Since for instance one point cannot be sent on two distinct points, such a map may not exist for any two distributions of masses. This problem were generalized much more later by Kantorovich [40] in a relaxed version. In his formalism, the supply and demand distributions (analogous with the piles of sand) were represented by two probability measures $\mu^\pm$ on $\mathbb{R}^d$ and the transport scheme is encoded by a probability measure Π on $\mathbb{R}^d \times \mathbb{R}^d$ (called *transport plan* or *transference plan*). Namely, $\Pi(A \times B)$ represents the amount of mass which is sent from A to B. Contrary to what happens for the Monge problem, in the Kantorovich problem, a given quantity of mass concentrated on a point can be spread on a large region. Given a function $c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^+$ representing the cost for moving from a point to another one, the total cost of a transport plan Π is given by

$$MK_c(\Pi) = \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x,y)\, \mathrm{d}\Pi(x,y).$$

Given a compact convex domain $X \subset \mathbb{R}^d$, the Monge-Kantorovich problem consists in minimizing MK_c over all transport plans Π connecting two probability measures μ^+ and μ^-. If $c(x,y) = |x-y|^p$ for some $p \in [1,+\infty)$, the minimal value of $MK_c =: MK_p$ induces a distance, called *Wasserstein distance*:

$$W_p(\mu^-,\mu^+) = \inf\left\{ \int_{X\times X} |x-y|^p \,\mathrm{d}\Pi(x,y) \ : \ \Pi \in \mathbf{\Pi}(\mu^-,\mu^+) \right\}^{\frac{1}{p}},$$

where $\mathbf{\Pi}(\mu^-,\mu^+)$ is the set of probability measures on X such that the image of the measure Π by the projection on the first (resp. second) variable is μ^- (resp. μ^+). The Monge-Kantorovich problem and the Wasserstein distances were intensively studied for theoretical interests as well as applications in many fields [8, 22, 32, 62]. We refer to [62] and [61] for further study in optimal transportation theory. In particular, one can show that W_p is a distance which induces the weak star topology on the set $\mathcal{P}(X)$ of probability measures on X.

More generally, it is possible to define the Wasserstein distance (still denoted by W_p) between nonnegative finite measures $\mu^\pm$ on X of equal mass $\int \mu^\pm =: \theta \geq 0$. To this aim, we proceed exactly in the same way as before, minimizing the total transport cost over the set $\Pi(\mu^-,\mu^+) = \theta\Pi(\nu^-,\nu^+)$, where $\nu^\pm := \theta^{-1}\mu^\pm \in \mathcal{P}(X)$. In particular, one has

$$W_p(\mu^-,\mu^+) = \theta^{\frac{1}{p}}\, W_p(\nu^-,\nu^+). \tag{1.1.5}$$

This easily implies that, if $(\mu_n^\pm)_{n\geq 0}$ are two sequences of nonnegative finite measures on X of equal mass $\theta_n := \int \mu_n^+ = \int \mu_n^-$, then $W_p(\mu_n^-,\mu_n^+) \underset{n\to\infty}{\longrightarrow} 0$ if and only if $\mu_n^+ - \mu_n^- \underset{n\to\infty}{\longrightarrow} 0$ weakly as measures.

When $\alpha = 1$, it turns out that the energy d^α matches with the Wasserstein distance for the Monge cost $c(x,y) = |x-y|$: $d^1 = W_1$. A very simple observation in that direction is that $m \to m^\alpha = m$ is not strictly concave but linear if $\alpha = 1$ so that branched structures are not encouraged anymore. In this case each unit of mass will follow a straight line between source and destination. That is why the information about the path covered by the mass is not needed: only the amount of mass which sent from some place to another one has to be known.

For general parameters $\alpha \in (0,1)$ and $p \in [1,+\infty)$, if $X = \overline{\Omega}$ where Ω is a convex and bounded domain, we have seen that d^α and W_p are distances which induces the weak star topology on $\mathcal{P}(X)$. Actually, we have a stronger property which is a comparison between the Wasserstein distances and the α-irrigation distances.

Proposition 1.1.10. *Assume that X is a compact and convex subset of $\mathbb{R}^d$. Let us fix $\alpha \in (0,1)$. Then, for every μ^-, $\mu^+ \in \mathcal{P}(X)$, one has*

$$W_{1/\alpha}(\mu^+,\mu^-) \leq d^\alpha(\mu^+,\mu^-) \leq C\, W_1(\mu^+,\mu^-)^{1-d(1-\alpha)},$$

for a constant $C > 0$ only depending on d, α and the diameter of X.

We refer to [51] for a proof of this property (see also [12], and [19] for an alternative proof). The question of the existence of such inequalities was raised by Cédric Villani in his report on Bernot Caselle's phd thesis ([9]). Note that all the Wasserstein distances W_p are trivially comparable since $W_p \le W_q$ whenever $p \le q$. Indeed, due to the Jensen inequality, for all probability measures $\Pi \in \mathbf{\Pi}(\mu^-, \mu^+)$, one has

$$\left(\int_{X\times X} |x-y|^p \,\mathrm{d}\Pi(x,y)\right)^{\frac{1}{p}} = \|x-y\|_{L^p(\Pi)} \le \|x-y\|_{L^q(\Pi)} = \left(\int_{X\times X} |x-y|^q \,\mathrm{d}\Pi(x,y)\right)^{\frac{1}{q}}.$$

1.2 Approximations of branched transportation: M_ε^α

For every measures to be irrigable, we make the following assumption (see Proposition 1.1.7):

$$1 - \frac{1}{d} < \alpha < 1. \tag{1.2.1}$$

We are interested in the following approximation of M^α which was introduced in [53]: for all $u \in \mathcal{M}_{div}(\Omega)$ and for all open subset $\omega \subset \Omega$,

$$M_\varepsilon^\alpha(u,\omega) := \begin{cases} \varepsilon^{-\gamma_1} \displaystyle\int_\omega |u(x)|^\beta \,\mathrm{d}x + \varepsilon^{\gamma_2} \int_\omega |\nabla u(x)|^2 \,\mathrm{d}x & \text{if } u \in H^1(\omega) \\ +\infty & \text{otherwise,} \end{cases} \tag{1.2.2}$$

for the specific choice of exponents γ_1, γ_2 and β (which will be justified after Theorem 1.2.2 below),

$$\gamma_1 = (d-1)(1-\alpha)\ , \quad \gamma_2 = 3 - d + \alpha(d-1) \quad \text{and} \quad \beta = \frac{2-2d+2\alpha d}{3-d+\alpha(d-1)}\ . \tag{1.2.3}$$

Note that inequality the $1 - 1/d < \alpha < 1$ implies that $0 < \beta < 1$. When $\omega = \Omega$, we simply write

$$M_\varepsilon^\alpha(u,\Omega) =: M_\varepsilon^\alpha(u).$$

We point out the 2-dimensional case where M_ε^α rewrites as

$$M_\varepsilon^\alpha(u) = \varepsilon^{\alpha-1} \int_\Omega |u(x)|^\beta \,\mathrm{d}x + \varepsilon^{\alpha+1} \int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x,$$

where $\beta = \frac{4\alpha-2}{\alpha+1}$.

Given two densities $f^+, f^- \in L^2_+(\Omega) := \{f \in L^2(\Omega) \ : \ f \ge 0\}$ such that $\int f^+ = \int f^-$, we are interested in minimizing $M_\varepsilon^\alpha(u)$ under the constraint $\nabla \cdot u = f^+ - f^-$:

$$\inf\left\{M_\varepsilon^\alpha(u) \ : \ u \in H^1(\Omega) \quad \text{and} \quad \nabla \cdot u = f^+ - f^-\right\}.$$

The classical theory of calculus of variation shows that this infimum is actually a minimum. A natural question that arises is then to understand the limit behavior for minimizers of these problems when ε goes to 0. A classical tool to study this kind of problems is the theory of Γ-convergence which was introduced by E. De Giorgi in [26]. For the definition and main properties of Γ-convergence, we refer to [24] and [18] (see also section of the introduction). In particular, if M_ε^α Γ-converges to some energy functional M_0^α and if (u_ε) is a sequence of minimizers for M_ε^α admitting a subsequence converging to u, then, u is a minimizer for M_0^α. By construction of M_ε^α, we expect that, up to a subsequence, M_ε^α Γ-converges to $c_0 M^\alpha$. In the two dimensional case, we have the following Γ-convergence theorem proved in [53]:

Theorem 1.2.1. *Assume that $d = 2$ and $\alpha \in (1/2, 1)$. Then, there exists a constant $c > 0$ such that $(M_\varepsilon^\alpha)_{\varepsilon>0}$ Γ-converges to cM^α in $\mathcal{M}_{div}(\Omega)$ when ε goes to 0.*

In the case $0 < \alpha \leq 1 - \frac{1}{d}$, the corresponding exponent β, given by (1.2.3), is negative. In [53], this difficulty was overcomed by replacing the potential $t \to t^\beta$ by a smooth potential $W(t)$ that behaves like t^β at infinity. In this way, the Γ-convergence was extended to the case of a any exponent $\alpha \in (0, 1)$. We point out the fact that the proof of Theorem 1.2.1 presented in [53] cannot be extended in higher dimension. In this thesis, we use a different proof based on the study of Cahn-Hilliard type models which allows to prove the Γ-convergence in every dimension. However, in higher dimension, we are not able to extend our result to the case $0 < \alpha \leq 1 - \frac{1}{d}$ anymore.

Theorem 1.2.2. *Let $d \geq 1$ be the dimension and assume that $1 - \frac{1}{d} < \alpha < 1$. Then, there exists $c_\beta > 0$ such that the energy functional $(M_\varepsilon^\alpha)_{\varepsilon>0}$ Γ-converges to $c_\beta M^\alpha$ as ε goes to 0. Moreover the constant $c_\beta > 0$ is given by (2.1.6) for $N = d - 1$.*

Heuristic Let us give a heuristic which shows why M_ε^α is an approximation of M^α (see [53]). Assume that μ^- (resp. μ^+) is a point source at S_1 (resp. S_2) with mass m. Then, it is clear that the optimal path for M^α between these two measures is the oriented edge $S = (S_1, S_2)$ of length l with a mass m flowing on it. We would like to approximate this structure, seen as a vector measure u concentrated on S, by some H^1 vector fields v which are more or less optimal for M_ε^α. What we expect is that v looks like a convolution of u with a kernel ρ depending on ε: $v = u * \rho_R$, where

$$\rho_R(x) = R^{-d}\rho(R^{-1}x) \tag{1.2.4}$$

for some fixed smooth and compactly supported radial kernel $\rho \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. Then the support of v is like a strip of width R around S so that $|v|$ is of the order of m/R^{d-1} and $|\nabla v|$ is of the order of m/R^d. This gives an estimate of $M_\varepsilon^\alpha(v)$ like

$$M_\varepsilon^\alpha(v) \simeq \varepsilon^{-\gamma_1} R^{d-1}(m/R^{d-1})^\beta l + \varepsilon^{\gamma_2} R^{d-1}(m/R^d)^2 l \quad . \tag{1.2.5}$$

With our choice for the exponents γ_1, γ_2 and β, the optimal choice for R is

$$R = \varepsilon^\gamma m^{\frac{1-\gamma}{d-1}}\,, \quad \text{where} \quad \gamma = \frac{2}{2d - \beta(d-1)} = \frac{\gamma_2}{d+1} \quad . \tag{1.2.6}$$

This finally leads to $M_\varepsilon^\alpha(v) \simeq m^\alpha$ as expected.

Note that Theorem 1.2.2 does not imply the Γ-convergence of $M_\varepsilon^\alpha + \mathbf{1}_{\nabla\cdot u = f^+ - f^-}$ to $M^\alpha + \mathbf{1}_{\nabla\cdot u = f^+ - f^-}$ even in two dimensions. Indeed, the Γ-convergence is stable under the addition of continuous functionals but not l.s.c. functionals. Consequently, we cannot deduce, from this theorem, the behavior of minimizers for M_ε^α under the divergence constraint. For instance, it is not clear that there exists a recovery sequence (u_ε), i.e. u_ε converges to u in $\mathcal{M}_{div}(\Omega)$ and $M_\varepsilon^\alpha(u_\varepsilon)$ converges to $M^\alpha(u)$ as $\varepsilon \to 0$, with prescribed divergence $\nabla \cdot u_\varepsilon = f^+ - f^-$. To this aim, we require some estimates on these energies and this is the purpose of the two last chapters of this part. As an application of these estimates, we will prove in chapter 4 the following result:

Theorem 1.2.3. *Let us fix $\mu^\pm$ two probability measures compactly supported on Ω and $\mu := \mu^+ - \mu^-$. Let $(f_\varepsilon)_{\varepsilon>0} \subset L^2(\Omega)$ be a sequence weakly converging to μ as measures when $\varepsilon \to 0$. Assume that the sequence $(f_\varepsilon)_{\varepsilon>0}$ satisfies*

$$\int_\Omega f_\varepsilon(x)\,\mathrm{d}x = 0 \quad \textit{and} \quad \varepsilon^{\gamma_2}\|f_\varepsilon\|_{L^2}^2 \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Let us define the functionals $\overline{M}_\varepsilon^\alpha$ and $\overline{M}^\alpha$ with divergence constraint:

$$\overline{M}_\varepsilon^\alpha(u) := M_\varepsilon^\alpha(u) + I_{\nabla\cdot u = f_\varepsilon} \quad \textit{and} \quad \overline{M}^\alpha(u) := M^\alpha(u) + I_{\nabla\cdot u = \mu^+ - \mu^-}.$$

Then, the functional sequence $(\overline{M}_\varepsilon^\alpha)_{\varepsilon>0}$ Γ-converges to $c_\beta \overline{M}^\alpha$ as $\varepsilon \to 0$ where c_β is the same constant which appears in Theorem 1.2.2.

Chapter 2

Γ-convergence in higher dimension, without divergence constraints

This chapter is devoted to the proof of Theorem 1.2.2 which generalizes Theorem 1.2.1 to every dimension. The proof of [53] uses a sort of linearization of the singular potential t^β around the origin whereas our proof is based on the analysis of Cahn-Hilliard models from droplets equilibrium which naturally appears in our Modica-Mortola type approximations M_ε^α. In this chapter, we ignore the divergence constraint, which will be taken into account in the two next chapters in order to prove Theorem 1.2.3.

2.1 Energy estimates on slices, the Cahn-Hilliard model

A classical tool for the study of energies defined on vector fields, representing a flow of some physical quantity for instance, consists in looking at the density of energy on each slice. More precisely, given some unit vector ν one can estimate the energy of some $u \in \mathcal{M}(\Omega)$ from below by the integral on $\mathbb{R}\nu$ of the flux of u across hyperplanes orthogonal to ν. This allows to consider only functionals defined on scalar functions in dimension $N := d - 1$. In this section, we are going to detail this procedure and then give a very quick overview of the Cahn-Hilliard energies for the equilibrium of small droplets.

Slicing method Let u be any compactly supported vector measure in $\mathcal{M}_{div}(\mathbb{R}^d)$. Take some $\nu \in S^{d-1} := \{x \in \mathbb{R}^d \ : \ |x| = 1\}$ and pick an orthonormal basis $(\nu_i)_{i=1,\dots,d}$ such that $\nu_1 = \nu$. For all $x \in \mathbb{R}^d$, we denote by $(y, x'_1, \dots, x'_{d-1})$ the coordinates of x in the basis $(\nu_i)_i$. For all $x = y\nu + \sum_{i=1}^{d-1} x'_i \nu_i \in \mathbb{R}^d$, let us consider $v(y, x') := [u(x)\cdot\nu]_+$ (positive part of $u \cdot \nu$) the flux of u through the hyperplane $H_y := y\nu + \mathrm{Span}(\nu_1, \dots, \nu_{d-1})$. Let us define the total flux of u across H_y by

$$\theta(y) := \int_{H_y} v(y,x')\,\mathrm{d}x' \quad \text{for} \quad x_1 \in \mathbb{R}.$$

Then $M_\varepsilon^\alpha(u)$ can be controlled from below by integrals on subintervals of $\mathbb{R}\nu$ as follows:

$$M_\varepsilon^\alpha(u) \geq \int_{\mathbb{R}} F_\varepsilon^\beta(v(y,\cdot))\,\mathrm{d}y \quad .$$

Here F_ε^β is a Cahn-Hilliard type energy defined for any nonnegative scalar function $v \in H^1(\mathbb{R}^N)$ by

$$F_\varepsilon^\beta(v) = \varepsilon^{-\gamma_1}\int_{\mathbb{R}^N} |v|^\beta + \varepsilon^{\gamma_2}\int_{\mathbb{R}^N} |\nabla v|^2 \quad , \tag{2.1.1}$$

where $N = d-1$. Then a natural estimate of the energy M_ε^α is given by

$$M_\varepsilon^\alpha(u) \geq \int_{\mathbb{R}} I_\varepsilon^\beta(\theta(y))\,\mathrm{d}y \quad ,$$

where $I_\varepsilon^\beta(\theta(x_1))$ stands for the minimal energy of $F_\varepsilon^\beta(v)$ under the volume constraint $\int v = \theta$:

$$I_\varepsilon^\beta(\theta) := \inf\left\{F_\varepsilon^\beta(v) \ : \ v \in H^1(\mathbb{R}^N \to \mathbb{R}^+), \quad \int_{\mathbb{R}^N} v = \theta\right\} \quad . \tag{2.1.2}$$

A solution to the problem (2.1.2) corresponds to the optimal profile in the minimization problem associated to E_ε. Such a solution is usually called Poiseuille profile in fluid mechanics.

Cahn-Hilliard fluids This kind of models for droplets equilibrium was studied by G. Bouchitté, C. Dubs and P. Seppecher in [15] for instance (see also [16]). These models come from the theory of Cahn-Hilliard fluids for small droplets, i.e. droplets which are very small compared to the domain size (see [21] for physical motivations). In [15], a more general situation than ours is considered. Namely, they consider general potentials $W : \mathbb{R}^+ \to \mathbb{R}^+$ which are equivalent to t^β for $\beta \in \mathbb{R}$ as t goes to $+\infty$. Yet, for simplicity and because of some technical difficulties, we restrict to the case $\beta > 0$. More precisely, they considered functionals defined for nonnegative scalar functions $v \in H^1(\mathbb{R}^N, \mathbb{R}^+)$ by

$$\int_{\mathbb{R}^N} W(v) + \varepsilon|\nabla v|^2\,\mathrm{d}x \quad , \tag{2.1.3}$$

where $N \geq 1$ is the dimension, $\varepsilon > 0$ is a small parameter, and W satisfied the following hypothesis:

1. W is continuous on $\mathbb{R}^+$,
2. $W(0) = 0$ and $W(t) > 0$ for $t > 0$,
3. there exists $t_0 > 0$ such that W is non-decreasing on $[0, t_0]$.
4. $\liminf\limits_{t\to 0^+} \frac{W(t)}{t} > 0$,
5. there exists $\beta \in \mathbb{R}$ such that $\frac{W(t)}{t^\beta} \xrightarrow[t\to\infty]{} 1$.

If $\beta < 0$ the study of the asymptotic behavior of energies like (2.1.3) is quite hard. For instance if $\beta < 0$ and $N \geq 3$, we are not able to prove that the limiting energy (in the sense of the Γ-convergence) is local. For this reason, from now on, we are going to assume that $\beta > 0$. In this case, an admissible potential W is given by

$$W(t) = t^\beta \quad \text{for all} \quad t > 0 \quad .$$

Note that (2.1.3) is nothing but the Modica-Mortola functional for two wells potentials W where the first well is at 0 and the other has been sent at infinity. Indeed 0 is a global minimizer of W and, somehow, $+\infty$ can be seen as a critical point. However the study of this kind of energies for singular potentials of this form turns out to be more difficult that for the classical Modica-Mortola functional, concerned with smooth potentials with two (finite) wells. In this chapter, we limit the discussion to the principal properties which will be useful later on, in the case $\beta > 0$.

The first thing to do is to find an equivalent of the energy (2.1.3) as ε goes to 0. In other words we want to renormalize the energy multiplying by some parameter λ_ε. This has already be done in the previous part for the energy M_ε^α. Applying the same heuristic, we find that the renormalization of (2.1.3) rewrites, up to replace ε by a power of it, as

$$F_\varepsilon^W(v) = \begin{cases} \int_{\mathbb{R}^N} \varepsilon^{-\gamma_1} W(v) + \varepsilon^{\gamma_2} |\nabla v|^2 \,\mathrm{d}x & \text{if } v \in H^1(\mathbb{R}^N) \text{ and } v \geq 0 \text{ a.e.} \\ +\infty & \text{otherwise,} \end{cases} \tag{2.1.4}$$

where γ_1 and γ_2 have been defined in (1.2.3): note that there is a unique $\alpha \in (1 - \frac{1}{d}, 1)$ such that the last equality in (1.2.3) holds. Since $\beta > 0$ one can choose $W(t) = t^\beta$ for $t > 0$. For the sake of simplicity, we will restrict to this case so that (2.1.4) simply rewrites as (2.1.1). Given $\theta \geq 0$ we consider the following minimization problem

$$\min \left\{ F_\varepsilon^\beta(v) \;:\; v \in H^1(\mathbb{R}^N, \mathbb{R}^+) \quad \text{and} \quad \int_{\mathbb{R}^N} v = \theta \right\}. \tag{2.1.5}$$

The infimum value of this problem has already been defined in (2.1.2) and was denoted by $I^\beta(\theta)$. Then two questions arises:

- Does a solution to the problem (2.1.5) exist ?
- How does the infimum value $I^\beta(\theta)$ depend on θ ?

Although it is easy to compute the dependance on θ of I^β by a scaling argument, the existence of a minimizer requires further attention.

To begin with, let us rescale the energy $F_\varepsilon^\beta(v)$. We want to reduce to an energy functional which does not depend anymore on ε and such that the volume constraint $\int v = \theta$ is replaced by $\int v = 1$. We make the following change of variables (rescaling):

$$v(x) = \theta R_{\theta,\varepsilon}^{-N} w(R_{\theta,\varepsilon}^{-1} x),$$

where

$$R_{\theta,\varepsilon} = \varepsilon^\gamma \theta^{\frac{1-\gamma}{N}} \quad \text{and} \quad \gamma = \frac{\gamma_2}{d+1} = \frac{\alpha+1}{3} \quad .$$

Then, the constraint $\int v = \theta$ turns into $\int w = 1$ and $F_\varepsilon^\beta(v) = \theta^\alpha F^\beta(w)$ where α is related to β through the formula (1.2.3) and

$$F^\beta(w) := \begin{cases} \int_{\mathbb{R}^N} |w|^\beta + |\nabla w|^2 \,\mathrm{d}x & \text{if } w \in H^1(\mathbb{R}^N) \text{ and } w \geq 0 \text{ a.e.} \\ +\infty & \text{otherwise.} \end{cases}$$

In particular, we have $I^\beta(\theta) = \theta^\alpha c_\beta$ where

$$c_\beta := \inf \left\{ \int_{\mathbb{R}^N} |w|^\beta + |\nabla w|^2 \ : \ w \in H^1(\mathbb{R}^N, \mathbb{R}^+) \quad \text{and} \quad \int_{\mathbb{R}^N} w = 1 \right\}. \tag{2.1.6}$$

The existence of an optimal profile w is given by

Lemma 2.1.1. *There exists a compactly supported and radially symmetric profile $w \in H_0^1(\mathbb{R}^N)$ solution of the minimization problem*

$$\min \left\{ \int_{\mathbb{R}^N} |w|^\beta + |\nabla w|^2 \ : \ w \in H^1(\mathbb{R}^N, \mathbb{R}^+) \quad \text{and} \quad \int_{\mathbb{R}^N} w = 1 \right\}. \tag{2.1.7}$$

Moreover, w is Lipschitz continuous on $\mathbb{R}^N$ and $\mathcal{C}^\infty$ inside its support, i.e. on the open set $(w > 0)$.

Remark 2.1.2. Note that the minimum value in (2.1.7) is related to the best constant in the Gagliardo-Nirenberg inequality $\int_{\mathbb{R}^N} |u| \leq C \left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^{\frac{t}{2}} \left(\int_{\mathbb{R}^N} |u|^\beta\right)^{\frac{1-t}{\beta}}$, determined by

$$\frac{1}{C} = \inf \left\{ \left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^{\frac{t}{2}} \left(\int_{\mathbb{R}^N} u^\beta\right)^{\frac{1-t}{\beta}} \ : \ u \in H^1_{loc}(\mathbb{R}^N, \mathbb{R}^+) \quad \text{and} \quad \int_{\mathbb{R}^N} u = 1 \right\},$$

where $t \in (0,1)$ is the only exponent satisfying $1 = (\frac{1}{2} - \frac{1}{d})t + \frac{1-t}{\beta}$.

We are going to prove the existence of a minimizer. In [29], it is proved that any critical point of the same energy is compactly supported. The main tool they used is the Pohozaev identity which is a consequence the following Euler-Lagrange equation associated to the minimization problem (2.1.7):

$$-\Delta w + \beta |w|^{\beta-2} w = \lambda \quad , \tag{2.1.8}$$

where $\lambda \in \mathbb{R}$ is the Lagrange multiplier associated to the volume constraint (see also [42] for the study of this kind of semilinear elliptic equations). Integrating (2.1.8) over the support of w, one can prove that $\lambda > 0$. Roughly speaking, the first term in the energy, $\int |w|^\beta$ with $0 < \beta < 1$, favors concentration (since the two "wells" are 0 and $+\infty$) whereas the Dirichlet term prefers diffuse functions. We have to show that there exists an equilibrium: w is large on some area and then decrease quickly to 0. In the following we only prove that any critical point, i.e. a solution of (2.1.8), which is radially symmetric is compactly supported.

Proof. First notice that there exists a finite energy configuration, i.e. $w \in H^1(\mathbb{R}^N, \mathbb{R})$ such that $F^\beta(w) < +\infty$. Indeed, all $\mathcal{C}^1$ functions which are compactly supported have finite energy. Let us take a minimizing sequence, i.e. $(w_n)_n$ such that

$$F^\beta(w_n) \to c_\beta \quad .$$

Now one can assume that the functions w_n are radially symmetric. Indeed if $w : \mathbb{R}^N \to \mathbb{R}$ is measurable then there exists a radially symmetric function w^*, namely the spherical rearrangement of w, such that $F^\beta(w^*) \leq F^\beta(w)$. We quickly remind how the spherical rearrangement is defined and why it decreases the energy. w^* is the unique radially symmetric and non-increasing function, i.e. $w^*(x) = f(|x|)$ with f non-increasing, such that

$$\forall \lambda > 0,\ \mathcal{L}^N(w^*(x) > \lambda) = \mathcal{L}^N(w(x) > \lambda) \quad ,$$

where $\mathcal{L}^N$ stands for the Lebesgue measure on $\mathbb{R}^N$. In the one-dimensional case, w^* is the generalized inverse of the repartition function of w. In general the spherical rearrangement can be expressed as follows

$$w^*(x) = \inf\{\lambda \in \mathbb{R} \ : \ \mathcal{L}^N(\{w(x) > \lambda\} < \alpha_N |x|^N)\} \quad ,$$

where α_N is the Lebesgue measure of the unit ball in $\mathbb{R}^N$. By construction, for all measurable function $F : \mathbb{R}^N \to \mathbb{R}^+$, one has $\int_{\mathbb{R}^N} F(w) = \int_{\mathbb{R}^N} F(w^*)$. Moreover, the Polya-Szego theorem (see [56], and [41] for a proof in every dimension) states that spherical rearrangement reduces the Dirichlet energy of w:

$$\int_{\mathbb{R}^N} |\nabla w^*|^2 \leq \int_{\mathbb{R}^N} |\nabla w|^2 \quad .$$

In particular, we have $F^\beta(w^*) \leq F^\beta(w)$ as announced. Since $(\nabla w_n)_n$ is bounded in $L^2(\mathbb{R}^N)$, one can assume that it weakly converges in $L^2(\mathbb{R}^N)$. Moreover, as $w_n \geq 0$ a.e. and $\int_{\mathbb{R}^N} w_n = 1$, $(w_n)_n$ is bounded in $L^1(\mathbb{R}^N)$. Thanks to the Poincaré-Wirtinger inequality, one deduces that $(w_n)_n$ is bounded in $H^1_{loc}(\mathbb{R}^N)$. Up to extraction, one can assume that $(w_n)_n$ weakly converges in $H^1_{loc}(\mathbb{R}^N)$. Let us call $w \in H^1_{loc}(\mathbb{R}^N)$ the limit. In particular, $(w_n)_n$ strongly converges to w in $L^1_{loc}(\mathbb{R}^N)$ and so w shares all the pointwise properties satisfied by the w_n, $n \geq 1$: $w(x) =: f(|x|)$ for some function $f = \mathbb{R}^+ \to \mathbb{R}^+$ non-increasing on $\mathbb{R}^+$. Note that f is also continuous since $w \in H^1_{loc}$. Now, the Fatou lemma and the weak convergence of ∇w_n yields

$$F^\beta(w) \leq \liminf_{n\to\infty} F^\beta(w_n) = c_\beta \quad .$$

In order to prove that w is a global minimizer it remains to prove that w satisfies the constraint $\int w = 1$. Indeed, from the Fatou Lemma we can only deduce that $\int w \leq 1$. One has to prove the strong convergence of w_n in $L^1(\mathbb{R}^N)$. Since w_n converges in L^1_{loc}, it is enough to prove that the sequence (w_n) is tight. Let $R > 0$. For all $n \geq 1$, since f is non increasing on $[0, R]$, one has $\|w_n\|_{L^\infty(\{x\,:\,|x|>R)} = f(R)$ and Markov's inequality yields $f(R) \leq \frac{1}{\omega_N R^N} \int_{\mathbb{R}^N} w_n = \frac{1}{\omega_N R^N}$ where ω_N stands for the volume of the unit ball in $\mathbb{R}^N$. Hence

$$\int_{|x|>R} w_n(x)\,\mathrm{d}x \leq f(R)^{1-\beta} \int_{\mathbb{R}^N} w_n^\beta(x)\,\mathrm{d}x \leq (\omega_N R^N)^{\beta-1} F^\beta(w_n) \leq \frac{C}{R^{N(1-\beta)}}$$

for some constant $C > 0$ non depending on n. This implies the claim since $1 - \beta > 0$.

Now, let us check the regularity of w: Lipschitz continuous and smooth inside its support. Since $w = w^*$, it can be written as $w(x) = f(|x|)$ for some function f : $\mathbb{R}^+ \to \mathbb{R}^+$ non-increasing and nonnegative. In particular the support of w is a ball

B_R of radius $R \in (0,\infty)$, centered at the origin: $B_R = \{x \in \mathbb{R}^N \ : \ |x| \leq R\}$. Since $\inf\{w(x) \ : \ x \in B_{R'}\} > 0$ for all $R' < R$, one can apply classical tools from elliptic theory (in particular bootstrapping methods) to see that w is $\mathcal{C}^\infty$ in the interior of the ball B_R. Moreover w is continuous on $\mathbb{R}^N$. Indeed, since $w \in H^1$, the function f is necessarily continuous and so is w. The Lipschitz continuity of f requires further attention. We first reinterpret the Euler-Lagrange equation (2.1.8) in terms of f. Easy computations yield

$$\forall r \in [0,R),\ -f''(r) - \frac{N-1}{r} f'(r) + \beta f(r)^{\beta-1} = \lambda \quad . \tag{2.1.9}$$

Now, multiplying (2.1.9) by f' gives

$$-\left[\frac{f'(r)^2}{2}\right]' - \frac{N-1}{r} f'(r)^2 + \left[f(r)^\beta\right]' = \lambda f'(r) \quad .$$

Since w has a global maximum at the origin, one has $\nabla w(0) = 0$ and so $f'(0) = 0$. Integrating the preceding equation then yields

$$\forall r \in [0,R),\ \frac{f'(r)^2}{2} + \int_0^r \frac{N-1}{s} f'(s)^2 \,\mathrm{d}s = f(r)^\beta - f(0)^\beta - \lambda f(r) + \lambda f(0) \quad .$$

Since f is bounded on $\mathbb{R}^+$ and since the second term of the left hand side in the preceding equation is nonnegative, we deduce that f' is bounded on $[0,R)$. In particular $f \in \mathrm{Lip}(\mathbb{R}^+)$ and w is Lipschitz continuous on $\mathbb{R}^N$.

Last of all, we prove that w is compactly supported, i.e. $R < \infty$. Assume by contradiction that $R = \infty$. Then $f \in \mathcal{C}^\infty([0,\infty))$ and multiplying (2.1.9) by r^{N-1} yields

$$\forall r \geq 0,\ [r^{N-1} f'(r)]' = (\beta f(r)^{\beta-1} - \lambda) r^{N-1} \quad .$$

Integrating this identity on $[0,r]$ for all $r \geq 0$ gives

$$\forall r \geq 0,\ r^{N-1} f'(r) = \int_0^r (\beta f(s)^{\beta-1} - \lambda) s^{N-1} \,\mathrm{d}s \quad .$$

Since $f(r) \underset{r\to\infty}{\longrightarrow} 0$ and $\beta - 1 < 0$, one has $f(r)^{\beta-1} \underset{r\to\infty}{\longrightarrow} +\infty$. Integrating on $[0,r]$, one can deduce that $\int_0^r (\beta f(s)^{\beta-1} - \lambda) s^{N-1} \,\mathrm{d}s \geq r^N$ for r large enough. Hence,

$$\exists r_0 > 0,\ \forall r \geq r_0,\ f'(r) \geq r$$

which is a contradiction with the fact that $f(r)$ goes to 0 as $r \to +\infty$. □

2.2 Application: proof of the lower bound

In this section, we prove the $\Gamma - \liminf$ property for the Γ-convergence of the functionals M_ε^α to M^α as $\varepsilon \to 0$ (Theorem 1.2.2). Namely, we will prove

Theorem 2.2.1. *Let $d \geq 2$ and $\Omega \subset \mathbb{R}^d$ some bounded open subset. Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence converging to u in $\mathcal{M}_{div}(\Omega)$. Then*

$$c_\beta M^\alpha(u) \leq \liminf_{\varepsilon\to 0} M^\alpha_\varepsilon(u_\varepsilon) \quad , \tag{2.2.1}$$

where c_β was defined in (2.1.6).

The main idea is to estimate the energy on each slice which is related to the scalar Cahn-Hilliard energy (see Lemma 2.2.2). Then we prove some compactness property which allows to pass to the limit. Finally we prove the regularity of the limiting configuration u and conclude the proof.

In all this section, we fix a sequence $(u_\varepsilon)_{\varepsilon>0} \subset \mathcal{M}_{div}(\Omega)$ such that

$$u_\varepsilon \xrightarrow[\varepsilon\to 0]{} u \quad \text{in } \mathcal{M}_{div}(\Omega).$$

In order to prove (2.2.1), one can assume that $M^\alpha_\varepsilon(u_\varepsilon) \leq C$ for some constant $C > 0$. Let $\mu_\varepsilon := \left\{\varepsilon^{\gamma_2}|\nabla u_\varepsilon|^2 + \varepsilon^{-\gamma_1}|u_\varepsilon|^\beta\right\} \mathrm{d}x$ be the energy density measure. Since μ_ε is bounded in the set of finite measures on Ω, one can assume that it weakly converges to some finite measure μ on Ω in the sense of measures:

$$\mu_\varepsilon \xrightarrow[\varepsilon\to 0]{*} \mu \quad . \tag{2.2.2}$$

Main estimate: We use the slicing method described in the previous part. As before, we fix $\nu \in S^{d-1} := \{x \in \mathbb{R}^d \ : \ |x| = 1\}$ and pick an orthonormal basis $(\nu_i)_{i=1,\dots,d}$ such that $\nu_1 = \nu$. For all $x \in \mathbb{R}^d$, we denote by $(y, x'_1, \dots, x'_{d-1})$ the system of coordinates associated to x in the basis $(\nu_i)_i$. Let

$$R := \{x_0 + y\nu + \sum_i x'_i\nu_i \in \Omega \ : \ -A \leq y \leq A \text{ and } -A' \leq x'_i \leq A' \text{ for } i = 1, \dots, d-1\}.$$

for some $x_0 \in \Omega$ and $A, A' > 0$. Let us assume that $\mu(\partial R) = |u|(\partial R) = 0$ which holds for all A, A' except countable many pairs (A, A'). Let $R_\delta = \{x \in R \ : \ \mathrm{dist}(x, R^c) > \delta\}$ for some $\delta > 0$. In order to work with compactly supported functions, we use cut-off functions: let $\xi \in \mathcal{C}^\infty_c(R)$ such that

$$0 \leq \xi(x) \leq 1, \ \xi(x) = 1 \quad \text{on } R_\delta \subset R, \quad \mathrm{Supp}(\xi) \subset R \quad \text{and} \quad \|\nabla\xi\|_\infty \leq \frac{2}{\delta} \quad .$$

For all $x = x_0 + y\nu + \sum_{i=2}^d x'_{i-1}\nu_i \in R$, let us consider

$$v_\varepsilon(y, x') := \xi(x)u_\varepsilon(x) \cdot \nu$$

the flux of u through the hyperplane $H_y := x_0 + y\nu + \mathrm{Span}\{(\nu_i)_{i=2,\dots,d}\}$. Let us define the total flux of u across H_y by

$$\theta_{\delta,\varepsilon}(y) := \int_{H_y} v_\varepsilon(y, x')\,\mathrm{d}x' \quad \text{for} \quad y \in \mathbb{R}.$$

Our main estimate is given in

Lemma 2.2.2. *The energy of u_ε on R is controlled from below by the integral of $|\theta_{\delta,\varepsilon}|^\alpha$, as follows*

$$M_\varepsilon^\alpha(u_\varepsilon, R) \geq c_\beta \int_{-A}^{A} |\theta_{\delta,\varepsilon}(y)|^\alpha \,\mathrm{d}y - r_{\delta,\varepsilon} \quad , \tag{2.2.3}$$

where $r_{\delta,\varepsilon}$ satisfies $\limsup\limits_{\delta\to 0} \limsup\limits_{\varepsilon\to 0} r_{\delta,\varepsilon} = 0$.

Proof. We start by estimating the difference between the energy of u and that of ξu. Since $\xi = 1$ on R_δ, one has

$$|M_\varepsilon^\alpha(u_\varepsilon, R) - M_\varepsilon^\alpha(\xi u_\varepsilon, R)| \leq M_\varepsilon^\alpha(u_\varepsilon, R \setminus R_\delta) + M_\varepsilon^\alpha(\xi u_\varepsilon, R \setminus R_\delta). \tag{2.2.4}$$

One has to estimate $M_\varepsilon^\alpha(\xi u_\varepsilon, R \setminus R_\delta) = \int_{R\setminus R_\delta} \varepsilon^{-\gamma_1}|\xi u_\varepsilon|^\beta + \varepsilon^{\gamma_2}|\nabla(\xi u_\varepsilon)|^2$. Since $\beta > 0$, $\|\xi\|_\infty \leq 1$ and $\|\nabla\xi\|_\infty \leq \frac{2}{\delta}$, one has

$$|\xi u_\varepsilon|^\beta \leq |u_\varepsilon|^\beta \quad \text{and} \quad |\nabla(\xi u_\varepsilon)|^2 \leq 2\left(|\nabla u_\varepsilon|^2 + \frac{4}{\delta^2}|u_\varepsilon|^2\right). \tag{2.2.5}$$

For the estimation of $\int_{R\setminus R_\delta} |u_\varepsilon|^2$, note that $R \setminus R_\delta = \{x \in R \ : \ \mathrm{dist}(x, \partial R) \leq \delta\}$ is a strip of size δ. In particular $R \setminus R_\delta$ is a finite union of squares of size length δ. Applying the Poincaré-Wirtinger inequality on one of these small squares $R' \subset R \setminus R_\delta$ yields

$$\int_{R'} \left| u_\varepsilon - \fint_{R'} u_\varepsilon \right|^2 \leq \frac{\delta^2}{2} \int_{R'} |\nabla u_\varepsilon|^2 \quad \text{and then} \quad \int_{R'} |u_\varepsilon|^2 \leq \delta^2 \int_{R'} |\nabla u_\varepsilon|^2 + \frac{2}{|R'|}\left(\int_{R'} u_\varepsilon\right)^2,$$

where $\fint_{R'} u_\varepsilon$ is the mean value of u_ε on R'. Taking the sum over all such cubes R' composing a partition of $R \setminus R_\delta$, one gets

$$\int_{R\setminus R_\delta} |u_\varepsilon|^2 \leq \delta^2 \int_{R\setminus R_\delta} |\nabla u_\varepsilon|^2 + C_\delta \left(\int_R |u_\varepsilon|\right)^2$$

for some constant $C_\delta > 0$ depending on R, d and δ. Then, the preceding equation, the definition of the energy density and (2.2.5) yield, for all $\varepsilon < 1$,

$$\begin{aligned}
M_\varepsilon^\alpha(\xi u_\varepsilon, R \setminus R_\delta) &\leq 2\, M_\varepsilon^\alpha(u_\varepsilon, R \setminus R_\delta) + \frac{8\,\varepsilon^{\gamma_2}}{\delta^2} \int_{R\setminus R_\delta} |u_\varepsilon|^2 \\
&\leq 2\, M_\varepsilon^\alpha(u_\varepsilon, R \setminus R_\delta) + 8\,\varepsilon^{\gamma_2} \int_{R\setminus R_\delta} |\nabla u_\varepsilon|^2 + C'_\delta\, \varepsilon^{\gamma_2} \left(\int_{R\setminus R_\delta} |u_\varepsilon|\right)^2 \\
&\leq 10\, M_\varepsilon^\alpha(u_\varepsilon, R \setminus R_\delta) + C'_\delta\, \varepsilon^{\gamma_2} \|u_\varepsilon\|_{L^1}^2
\end{aligned}$$

for some constants C'_δ non depending on ε. Since the sequence $(u_\varepsilon)_{\varepsilon>0}$ is bounded in L^1 by assumption, the preceding equation and (2.2.4) yield

$$|M_\varepsilon^\alpha(u_\varepsilon, R) - M_\varepsilon^\alpha(\xi u_\varepsilon, R)| \leq 10\, M_\varepsilon^\alpha(u_\varepsilon, R \setminus R_\delta) + C'''_\delta \varepsilon^{\gamma_2}, \tag{2.2.6}$$

where the constant $C'''_\delta > 0$ depends on δ, R, d and the sequence $(u_\varepsilon)_\varepsilon$ but does not depend on ε. Let $r_{\delta,\varepsilon} := 10 M_\varepsilon^\alpha(u_\varepsilon, R \setminus R_\delta) + C'''_\delta \varepsilon^{\gamma_2}$. Now, the weak convergence of the energy density, that is (2.2.2), and the assumption $\mu(\partial R) = 0$, give

$$\limsup_{\delta\to 0} \limsup_{\varepsilon\to 0} r_{\delta,\varepsilon} \leq 10 \limsup_{\delta\to 0} \mu(R \setminus R_\delta) = 0.$$

Thanks to (2.2.6), it is enough to estimate $E_\varepsilon^\alpha(\xi u_\varepsilon)$ from below (instead of $E_\varepsilon^\alpha(u_\varepsilon)$). We recall the notation $v_\varepsilon(y,x') = \xi(x)u_\varepsilon(x)\cdot\nu$. As we noticed in the previous section, one has

$$\begin{aligned} M_\varepsilon^\alpha(\xi u_\varepsilon) &= \int_\Omega \varepsilon^{\gamma_2}|\nabla u_\varepsilon|^2 + \varepsilon^{-\gamma_1}|u_\varepsilon|^\beta \,\mathrm{d}x \\ &\geq \int_{-A}^{A}\int_{H_y} \varepsilon^{\gamma_2}|\nabla'(v_\varepsilon)_+|^2 + \varepsilon^{-\gamma_1}|(v_\varepsilon)_+|^\beta \,\mathrm{d}x'\,\mathrm{d}y \\ &\geq \int_{-A}^{A} F^\beta\left(\int_{H_y}(v_\varepsilon)_+(y,x')\right)\mathrm{d}y \quad , \end{aligned}$$

where ∇' stands for the gradient with respect to the $d-1$ last variables, i.e. x'. Now, we introduce the notation

$$\theta_{\delta,\varepsilon}(y) := \int_{H_y} v_\varepsilon(y,x')\,\mathrm{d}x' \quad .$$

If $\theta_{\delta,\varepsilon}(y)\geq 0$, one has $\int_{H_y}(v_\varepsilon)_+ \geq \theta_{\delta,\varepsilon}(y) = \int_{H_y}(v_\varepsilon)_+ - \int_{H_y}(v_\varepsilon)_-$ and so

$$F^\beta\left(\int_{H_y}(v_\varepsilon)_+(y,x')\,\mathrm{d}x'\right) \geq F^\beta(\theta_{\delta,\varepsilon}(y)) = c_\beta\theta_{\delta,\varepsilon}(y)^\alpha.$$

If $\theta_{\delta,\varepsilon}(y)\leq 0$ we use the negative part of v_ε instead of the positive part and, at the end, we get

$$M_\varepsilon^\alpha(\xi u_\varepsilon) \geq c_\beta\int_{-A}^{A}|\theta_{\delta,\varepsilon}(y)|^\alpha\,\mathrm{d}y \quad . \quad \square$$

Limit as $\varepsilon\to 0$ Now we want to pass to the limit in (2.2.3). Because of the non-convex term $\int|\theta_{\delta,\varepsilon}|^\alpha$, we need strong compactness on $\theta_{\delta,\varepsilon}$. Fortunately, we are going to prove a BV bound on $\theta_{\delta,\varepsilon}$ which guarantees pointwise convergence:

Lemma 2.2.3. *The sequence $(\theta_{\delta,\varepsilon})_{\varepsilon>0}$ is bounded in the space of bounded variation functions, $BV([-A,A])$.*

Remark 2.2.4. Even if the proof used in [53] is different, Lemma 2.2.3 is also a cornerstone in their proof of the $\Gamma-\liminf$ property in dimension 2.

Proof. Let $\varphi\in\mathcal{C}^1([-A,A])$ such that $\varphi(-A)=\varphi(A)=0$. Up to a change coordinates, using a translation and a rotation, one can assume that $x_0=0$ and $\nu=e_1$, first vector of the canonical basis of $\mathbb{R}^d$. Then $R=[-A,A]\times[-A',A']^{d-1}$ and $v_\varepsilon(x)=\xi(x)u_\varepsilon(x)\cdot e_1$

for all $x = (y, x') \in \mathbb{R}^d$. Now we can compute

$$\begin{aligned}
\int_{-A}^{A} \theta_{\delta,\varepsilon}(y)\varphi'(y)\,\mathrm{d}y &= \int_{-A}^{A}\int_{H_y} v_\varepsilon(y,x')\varphi'(y)\,\mathrm{d}x'\,\mathrm{d}y\\
&= \int_\Omega \xi(x)u_\varepsilon(x)\cdot e_1\varphi'(x_1)\,\mathrm{d}x\\
&= \int_\Omega \xi(x)u_\varepsilon(x)\cdot \nabla_x\varphi(x_1)\,\mathrm{d}x\\
&= \int_\Omega \nabla\cdot\{\xi(x)u_\varepsilon(x)\}\varphi(x_1)\,\mathrm{d}x\\
&= \int_\Omega \left[\xi(x)\nabla\cdot\{u_\varepsilon(x)\} + \nabla\xi(x)\cdot u_\varepsilon(x)\right]\varphi(x_1)\,\mathrm{d}x\\
&\le \|\varphi\|_\infty \left\{\int_\Omega |\nabla\cdot u_\varepsilon|\,\mathrm{d}x + \|\nabla\xi\|_\infty\int_\Omega |u_\varepsilon|\,\mathrm{d}x\right\}.
\end{aligned}$$

Since u_ε converges in $\mathcal{M}_{div}$, both $(u_\varepsilon)_\varepsilon$ and $(\nabla\cdot u_\varepsilon)_\varepsilon$ are bounded as measures on Ω and the result follows. □

From now on we assume that x_0 is at the origin and $\nu = e_1$ so that $R = [-A, A] \times [-A', A']^{d-1}$. There is no lack of generality since one can come down to this case making a change of variables. Let denote by $\theta_\delta(y)$ the limit of $\theta_{\delta,\varepsilon}$ in $BV([-A, A])$. Since u_ε weakly converges to $u \in \mathcal{M}_{div}(\Omega)$, θ_δ is the density of the projection of $\xi u\cdot\nu = \xi u\cdot e_1 = \xi u_1$ on $\mathbb{R}\nu$:

$$\theta_\delta(y)\mathcal{L}^1 = (\Pi_y)_\#(\xi u_1),$$

where Π_y stands for the projection on the first variable y, $\Pi_y((y, x')) = y$, and $\mathcal{L}^d$ is the d-dimensional Lebesgue measure.

Both Lemma 2.2.2 and Lemma 2.2.3 imply, taking the lim inf when $\varepsilon \to 0$ in (2.2.3), that

$$\mu(R) \ge c_\beta \int_{-A}^{A} |\theta_\delta(y)|^\alpha\,\mathrm{d}y - \limsup_{\varepsilon\to 0} r_{\delta,\varepsilon} \quad , \tag{2.2.7}$$

where we have used the Fatou lemma and the weak convergence of the energy density μ_ε to μ. Note that ξ actually depends on δ: $\xi = \xi_\delta$. However, it is easy to pass to the limit when $\delta \to 0$ and get rid of the cut-off function ξ. Indeed, since $|u|(\partial R) = 0$, $\xi_\delta u_1$ strongly converges to $u_1\mathbf{1}_R$ as measures when $\delta \to 0$. Thus $\theta_\delta(y)\mathcal{L}^d = (\Pi_y)_\#(\xi_\delta u_1)$ strongly converges as well. Since $(\theta_\delta)_\delta \in L^1$, this sequence actually converges in L^1. Let denote by $\theta \in L^1$ the limit. Hence

$$\mu(R) \ge c_\beta \int_{-A}^{A} |\theta(y)|^\alpha\,\mathrm{d}y \quad .$$

Note that $\theta(\cdot)\mathcal{L}^d = (\Pi_y)_\#(u_1)$. More generally, we have shown that for all interval $I \subset \mathbb{R}$ and a.e. hypercube $R' \subset \mathbb{R}^{d-1}$ such that $I \times R' \subset \Omega$, one has

$$\mu(I \times R') \ge c_\beta \int_I |\theta(R', y)|^\alpha\,\mathrm{d}y \quad , \tag{2.2.8}$$

where $\theta(R', \cdot)$ is the density of $((\Pi_y)_{|R'})_\#(u_1)$ with respect to $\mathcal{L}^1$.

Rectifiability of u We want to prove that (2.2.8) implies the one-dimensional rectifiability of u. It is enough to prove the rectifiablity when $\Omega = I_0 \times R'_0$ for some interval I_0 and some hypercube R'_0.

We first make inequality (2.2.8) more explicit by interpreting $\theta(R', y)$ as a slice of the measure u_1. Let $\nu_0 \in \mathcal{M}(I_0)$ be the projection of $|u_1|$ on the first coordinate axis. We can disintegrate the measure u_1 with respect to ν_0 and get some signed measures $(\nu_y)_{y\in\mathbb{R}}$ on R'_0 such that $\|\nu_y\|(R'_0) = 1$. We remind that the disintegration is characterized by

$$\forall\varphi \in \mathcal{C}(\Omega),\ \int_\Omega u_1(x)\varphi(x)\,\mathrm{d}x = \int_{I_0}\int_{R'_0} \varphi(y, x')\,\mathrm{d}\nu_y(x')\,\mathrm{d}\nu_0(y) \quad .$$

Note that ν_0, as the projection of u_1 (without $|\cdot|$), is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^1$: $\nu_0 \ll \mathcal{L}^1$. Indeed, let $I \subset I_0$ be any negligible set for the Lebesgue measure and $R' \subset R'_0$ some hypercube such that its vertices have rational coordinates. By definition one has

$$((\Pi_y)_{|R'})_\# u_1 = \theta(R', y)\,\mathrm{d}y = \nu_y(R')\,\mathrm{d}\nu_0(y) \quad .$$

Since $\mathcal{L}^1(I) = 0$, one has $\int_{I'} \nu_y(R')\,\mathrm{d}\nu_0(y) = 0$ for all $I' \subset I$. Let $\nu_0 \llcorner I$ be the restriction of ν_0 to I. Then, if y is a Lebesgue point for $y \to \nu_y(R')$, one has $\nu_y(R') = 0$. Since $\nu_0 \llcorner I$-a.e. point in I is Lebesgue point and since R' have been chosen in a countable class (hypercubes such that its vertices have rational coordinates), one deduces that

$$\nu_0 \llcorner I - a.e.,\ \nu_y(R') = 0 \quad \text{for all} \quad R' \quad .$$

Since $\|\nu_y\| = 1$, ν_y does not vanish and so $\nu_0(I) = 0$ and $\nu_0 \ll \mathcal{L}^1$ as we claimed. Let $m(\cdot)$ be the density of ν_0 with respect to $\mathcal{L}^1$. Then, for all hypercube $R' \subset R$, $\theta(R', y) = \nu_y(R')m(y)$ a.e. and inequality (2.2.8) rewrites

$$\mu(I \times R') \geq c_\beta \int_I |\nu_y(R')|^\alpha m(y)^\alpha\,\mathrm{d}y \quad . \tag{2.2.9}$$

Now, we prove that (2.2.9) implies that u is one-dimensional rectifiable. The proof follows that of E. Oudet and F. Santambrogio in [53]. We use a rectifiability theorem proved by H. Federer and B. White [31, 63] which states that u is one-dimensional rectifiable if and only if almost every $(d-1)$-dimensional slices of u perpendicular to some coordinates axes are atomic measures. The last property concerning the slices of u is an easy application of the concentration compactness principal which was introduced by P-L Lions (see [46]). Since our situation is not very usual, we make a proof specific to our vectorial framework. We use the following lemma which is very classic in the concentration compactness principle:

Lemma 2.2.5. *Let μ be a positive Radon measure and ν a signed Radon measure on some open set $\Omega \subset \mathbb{R}^d$, $d \geq 1$. Assume that there exists $\alpha \in (0, 1)$ such that for all Borelian set B, one has*

$$\mu(B) \geq |\nu(B)|^\alpha \,.$$

Then ν is an atomic measures, i.e. there exists $(x_i, m_i)_{i\geq 1} \in \{\Omega \times \mathbb{R}\}^{\mathbb{N}^*}$ such that

$$\nu = \sum_{i=1}^{\infty} m_i \delta_{x_i} \quad .$$

Proof. The assumption implies in particular that ν is absolutely continuous with respect to μ. By the Radon-Nikodym, ν has a density with respect to μ. Moreover this density, denoted by $\frac{\mathrm{d}\mu}{\mathrm{d}\nu}$ can be computed as

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x) = \lim_{r\to 0} \frac{\nu(B(x,r))}{\mu(B(x,r))}$$

for μ-a.e. $x \in \Omega$, where we take the convention $\frac{\nu(B(x,r))}{\mu(B(x,r))} = 0$ whenever $\mu(B(x,r)) = 0$. Since, by assumption, $\frac{\nu(B(x,r))}{\mu(B(x,r))} \leq |\nu(B(x,r))|^{1-\alpha}$, we deduce that for μ-a.e. $x \in \Omega$,

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x) \leq \limsup_{r\to 0} |\nu(B(x,r))|^{1-\alpha} = |\nu(\{x\})|^{1-\alpha} \quad .$$

In particular $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)$ vanishes everywhere except possibly for countably many points x_i, i.e. ν is atomic. □

Now the rectifiability of almost every slices of u_1 follows from the following Lemma

Lemma 2.2.6. *Assume that* (2.2.9) *is satisfied. Then for ν_0-a.e. $y \in I_0$, ν_y is atomic,i.e.*

$$\nu_y = \sum_{i=1}^{\infty} m_i \delta_{x_1}$$

for some $m_i \in \mathbb{R}$ and $x_i \in R'$.

Proof. Let μ_0 be the projection of μ on the first coordinate axis and let disintegrate μ with respect to μ_0: one gets a family of probability measures μ_y for $y \in I_0$. As before, when we proved that $\nu_0 \ll \mathcal{L}^1$, one can prove that $m(y)^\alpha \,\mathrm{d}y$ is absolute continuous with respect to μ_0 as a direct consequence of (2.2.9). Let write η its density: $m(y)^\alpha \,\mathrm{d}y = \eta(y)\,\mathrm{d}\mu_0(y)$. Then (2.2.9) is equivalent to

$$\int_I \mu_y(R')\,\mathrm{d}\mu_0(y) \geq c_\beta \int_I |\nu_y(R')|^\alpha \eta(y)\,\mathrm{d}\mu_0(y).$$

As before, considering Lebesgue points for the measure μ_0, one can deduce that for μ_0-a.e. $y \in I_0$,

$$\mu_y(R') \geq c_\beta\, \eta(y)\, |\nu_y(R')|^\alpha \quad \text{for all } R'.$$

Note that, since $\mathrm{d}\nu_0(y) = m(y)\,\mathrm{d}y$ and $m(y)^\alpha \,\mathrm{d}y = \eta(y)\,\mathrm{d}\mu_0(y)$, one has $\eta(y) > 0$ for ν_0-a.e. $y \in I_0$. Now Lemma 2.2.6 follows from Lemma 2.2.5. □

We have proved that almost all slices of u_1 perpendicular to the first coordinate axis are atomic. Since the cases of other coordinate axis are similar, we conclude from White's criterium that u is one-dimensional rectifiable.

End of the proof of Theorem 2.2.1 Since u is one-dimensional rectifiable, it can be written as $u = U(M, \theta, \xi)$ where $U(M, \theta, \xi)$ has been defined in the introduction: it is the rectifiable vector measure $u = \theta\xi \cdot \mathcal{H}^1 \llcorner M$ with density $\theta\xi$ with respect to the $\mathcal{H}^1$–Hausdorff measure on the rectifiable set M. The real multiplicity is a measurable function $\theta : M \to \mathbb{R}^+$ and the orientation $\xi : M \to S^{d-1} \subset \mathbb{R}^d$ is such that $\xi(x)$ is tangential to M for $\mathcal{H}^1$-a.e. $x \in M$.

Now the $\Gamma - \liminf$ property is equivalent to the inequality $\mu \geq c_\beta \theta^\alpha \mathcal{H}^1 \llcorner M$ or equivalently,

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^1 \llcorner M} \geq c_\beta \theta^\alpha \quad \text{a.e.}$$

which is a direct consequence of (2.2.8). Indeed, for $\mathcal{H}^1 \llcorner M - a.e.\ x_0$, one can choose small hyperrectangles R centered at x_0 and oriented as the tangent vector to M and get the inequality in the limit when the size of R goes to 0.

2.3 Proof of the upper bound

This section is devoted to the proof of the upper bound of the Γ-convergence result, Theorem 1.2.2. Namely, we will prove the following theorem:

Theorem 2.3.1. *Let $d \geq 2$ and $\Omega \subset \mathbb{R}^d$ a bounded open set. Let $u \in \mathcal{M}_{div}(\Omega)$ a measure with finite energy, i.e. $M^\alpha(u) < +\infty$. Then there exists a sequence $(u_\varepsilon)_{\varepsilon > 0} \subset \mathcal{M}_{div}(\Omega)$ converging to u as $\varepsilon \to 0$ such that*

$$M^\alpha_\varepsilon(u_\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} c_\beta M^\alpha(u).$$

In view of the analysis of Cahn-Hilliard fluids in section 2.1, the proof of this theorem is very easy. Indeed, first assume that u is of the form $u = \theta_0 \tau_0 \mathcal{H}^1 \llcorner S$ for some segment S and $\theta_0 > 0$ where τ_0 is a unit vector tangential to S. Then it is enough to build u_ε from u by spreading the mass on a strip around S in such a way that it matches with the optimal profile for the problem (2.1.2). Then we conclude by density to obtain a recovery sequence for every $u \in \mathcal{M}_{div}(\Omega)$. This approximation procedure is due to some works of Q. Xia (see [64]). In particular, it is shown that the set of measures concentrated on finite graphs is dense in energy for M^α (see Proposition 1.1.6). This was used in [53] to prove the Γ-convergence of M^α_ε toward M^α in dimension 2. The proof easily extend to higher dimension:

Proof of Theorem 2.3.1. Due to Propostion 1.1.6, one can assume that $u = u_G$ is concentrated on a weighted oriented graph G. Let start with the case where $u = \theta\tau\mathcal{H}^1 \llcorner S$ is concentrated on a single segment $S \subset \Omega$, say $S = \{te_1 = (t, 0, \ldots, 0) \in \mathbb{R}^d \ : \ 0 \leq t \leq 1\}$ oriented by its tangential vector $\tau = e_1$. Let w be an optimal profile for the problem 2.1.7 in dimension $N := d - 1$. We remind the definition of $R_{\theta,\varepsilon} = \varepsilon^\gamma \theta^{\frac{1-\gamma}{N}}$ where $\gamma = \frac{\gamma_2}{d+1}$. Then v_ε defined by $v_\varepsilon(x) = \theta R_{\theta,\varepsilon}^{-N} w(R_{\theta,\varepsilon}^{-1} x)$ is optimal for the minimization problem (2.1.5), that is:

$$F^\beta_\varepsilon(v_\varepsilon) = c_\beta \theta^\alpha = \inf\left\{ F^\beta_\varepsilon(v) \ : \ v \in H^1(\mathbb{R}^N, \mathbb{R}^+) \quad \text{and} \quad \int_{\mathbb{R}^N} v = \theta \right\}.$$

Assume that ε is small enough so that $S_\varepsilon := \{x \in \mathbb{R}^d \ : \ \operatorname{dist}(x,S) \le \operatorname{Diam}(\operatorname{Supp}(v_\varepsilon))\} \subset \Omega$. Then, one can define a recovery sequence u_ε for $x = (x_1, x') \in \mathbb{R}^d$ by

$$u_\varepsilon(x) = \begin{cases} v_\varepsilon(x')e_1 & \text{if } 0 \le x_1 \le 1, \\ v_\varepsilon(|x|)e_1 & \text{if } x_1 < 0, \\ v_\varepsilon(|x-e_1|)e_1 & \text{if } x_1 > 1 \ . \end{cases}$$

Then u_ε belongs to $H^1_0(\Omega)$ and converges to u in $\mathcal{M}_{div}(\Omega)$. Moreover, $M^\alpha_\varepsilon(u_\varepsilon,\omega) = \int_S F^\beta_\varepsilon(v_\varepsilon)\,\mathrm{d}x_1 = c_\beta\theta^\alpha = c_\beta M^\alpha(u)$ where $\omega = \{(x_1,x') \ : \ 0 \le x_1 \le 1, \ x' \in \mathbb{R}^{d-1}\}$. Hence, it remains to estimate the energy of u_ε out of ω. One can prove that it is negligeable as $\varepsilon \to 0$. Indeed, let $B = B(O, R_{\varepsilon,\theta})$ the ball centered at the origin of radius $R_{\theta,\varepsilon}$. Then $M^\alpha_\varepsilon(u_\varepsilon,\omega^c) \le 2M^\alpha_\varepsilon(u_\varepsilon,B)$. Moreover, one has

$$\begin{aligned} M^\alpha_\varepsilon(u_\varepsilon,B) &= \int_B \varepsilon^{\gamma_2}|\nabla u_\varepsilon|^2 + \varepsilon^{-\gamma_1}|u_\varepsilon|^\beta \,\mathrm{d}x, \\ &\le |B|\{\varepsilon^{\gamma_2}\|\nabla v_\varepsilon\|^2_{L^\infty} + \varepsilon^{-\gamma_1}\|v_\varepsilon\|^\beta_{L^\infty}\} \\ &\le C\varepsilon^{d\gamma}\{\varepsilon^{\gamma_2-2\gamma(N+1)} + \varepsilon^{-\gamma_1-N\beta\gamma}\} \\ &\le 2C\varepsilon^\gamma. \end{aligned}$$

for some constant $C > 0$ independant of ε, where the following elementary computations on exponents γ, γ_1, γ_2 and β have been used:

$$\begin{cases} d\gamma + \gamma_2 - 2\gamma(N+1) = \gamma & , \\ d\gamma - \gamma_1 - N\beta\gamma = \gamma & . \end{cases}$$

Since $\varepsilon^\gamma \underset{\varepsilon\to 0}{\longrightarrow} 0$, we deduce that $(u_\varepsilon)_{\varepsilon>0}$ is a recovery sequence. In general, if u is concentrated on a finite graph, $u = \sum_{i=1}^I \theta_i\tau_i\mathcal{H}^1 \llcorner S_i$, then similar computations give the same result. Indeed, we just approximate each $u_i := \theta_i\tau_i\mathcal{H}^1 \llcorner S_i$ by some u^i_ε as before and then take the sum. In this way we get a sequence $(u_\varepsilon)_{\varepsilon>0}$ converging to u. Moreover, on the set where the u^i_ε are disjoint, the energy of u_ε is of the order of $M^\alpha(u)$ while it is negligible on the set where at least the support of two of the u^i_ε intersect. Thus $(u_\varepsilon)_{\varepsilon>0}$ is a recovery sequence. □

Remark 2.3.2. Although we have used Lemma 2.1.1 in the proof, we could do without it. Indeed, instead of an optimal profile w, one could have used a minimizing sequence w_ε.

The proof of the Γ-convergence of functionals M^α_ε is now complete. Note that Theorem 1.2.2 does not take into account any divergence constraint. Thus, we do not know from this theorem if $\overline{M}^\alpha_\varepsilon$ (defined with a divergence constraint) Γ-converges to $\overline{M}^\alpha$ as well. It is clear that the $\Gamma - \liminf$ property still is true when adding a constraint. However, this is not evident for the $\Gamma - \limsup$ property. Indeed, the divergence constraint could be violated by the recovery sequence $(u_\varepsilon)_{\varepsilon>0}$ obtained in the proof of Theorem 2.3.1. Namely u_ε is not divergence-free anymore at the junctions of the graph where u is concentrated. That is why we have to correct u_ε around the "node set" (union of the junctions). In the following chapter, we prove some uniform estimates allowing to lead this "divergence-correction" procedure. This will be our main tool to prove the Γ-convergence property under divergence constraint.

Chapter 3

Uniform estimates on the functionals M_ε^α

This chapter is for a large part contained in the article [50].

In section 3.1 we define some pseudo-distances d_ε^α induced by M_ε^α, defined by analogy with the branched transportation distances d^α. The longest part of this chapter, section 3.2, is devoted to a local estimate which gives a bound on the minimum value $d_\varepsilon^\alpha(f^+,f^-) := \min\{M_\varepsilon^\alpha(u)\ :\ \nabla\cdot u = f\}$ depending on $\|f\|_{L^1}$, $\|f\|_{L^2}$ and $\operatorname{diam}(\Omega)$ (see Proposition 3.2.2 page 60). In section 3.3, we deduce a comparison between d_ε^α and the Wasserstein distance with an "error term" involving the L^2 norm of $f^+ - f^-$. As an application of this inequality, in the last chapter of the first part, we will prove the Γ-convergence result which was lacking in [53], of functionals $\overline{M}_\varepsilon^\alpha$ to $\overline{M}^\alpha$ (with a divergence constraint on $\nabla\cdot u$): this answers the Open question 1 in [59, 53] and validates their numerical method.

3.1 Distances d_ε^α induced by M_ε^α

We remind our irrigability assumption $1-1/d<\alpha<1$ which allows to consider the induced distance d^α defined by minimization of the branched transportation energy M^α (see section 1.1.3). In the same way, we define the pseudo-distances d_ε^α induced by M_ε^α as follows:

$$d_\varepsilon^\alpha(f^+,f^-) = \inf\left\{M_\varepsilon^\alpha(u)\ :\ u\in H^1(\mathbb{R}^d)\quad\text{such that}\quad \nabla\cdot u = f^+-f^-\right\},\qquad(3.1.1)$$

where $f^+,f^-\in L^2_+(\Omega)$ satisfy $\int_\Omega f^+ = \int_\Omega f^-$. Although d^α is a distance, it is not the case for d_ε^α which does not satisfy the triangular inequality. Actually, because of the second term involving $|\nabla u|^2$, M_ε^α is not subadditive. However, for $u_1,\dots,u_n$ in $\mathcal{M}_{div}(\Omega)$, the inequality $|\nabla u_1+\cdots+\nabla u_n|^2\le n\{|\nabla u_1|^2+\cdots+|\nabla u_n|^2\}$ implies

$$M_\varepsilon^\alpha\left(\sum_{i=1}^n u_i\right)\le n\sum_{i=1}^n M_\varepsilon^\alpha(u_i).$$

In particular, d_ε^α is a pseudo-distance in the sense that the three properties in the following proposition are satisfied:

Proposition 3.1.1. *Let f^+, f^- and $f_1,\dots,$ f_n be L^2 densities, i.e. L^2 nonnegative functions whose integral is equal to 1. Then one has*

1. $d_\varepsilon^\alpha(f^+, f^-) = 0$ *implies* $f^+ = f^-$,
2. $d_\varepsilon^\alpha(f^+, f^-) = d_\varepsilon^\alpha(f^-, f^+)$,
3. $d_\varepsilon^\alpha(f_0, f_n) \le n\,\big[d_\varepsilon^\alpha(f_0, f_1) + d_\varepsilon^\alpha(f_1, f_2) + \cdots + d_\varepsilon^\alpha(f_{n-1}, f_n)\big]$.

3.2 Local estimate

We remind our assumption (1.2.1) which insures that $d^\alpha(\mu^+, \mu^-)$ is always finite. Our goal is to prove that d_ε^α enjoys a property similar to the following one.

Proposition 3.2.1. *Let $Q_0 = (0, L)^d \subset \mathbb{R}^d$ be a cube of side length $L > 0$. There exists some constant $C > 0$ only depending on d and α such that for all nonnegative Borel finite measure μ of total mass $\theta > 0$,*

$$d^\alpha(\mu, \theta\delta_0) \le C\ \theta^\alpha L,$$

where δ_0 is the Dirac measure at the point c_{Q_0}, the center of Q_0.

Since $d_\varepsilon^\alpha(f^+, f^-)$ is only defined on L^2 functions $f^\pm$, to do so, we first have to replace $\theta\delta_0$ by some kernel which concentrates at the origin when ε goes to 0. Let $\rho \in \mathcal{C}_c^1(B, \mathbb{R}^+)$ be a radial nonnegative function such that $\int_{\mathbb{R}^d} \rho = 1$, where $B \subset \mathbb{R}^d$ is the unit ball centered at the origin, and define $\rho_{\theta,\varepsilon} := \rho_R$ as in (1.2.4), where

$$R =: R_{\theta,\varepsilon} = \varepsilon^\gamma \theta^{\frac{1-\gamma}{d-1}}.$$

Here, we recall that R and $\gamma = \frac{\gamma_2}{d+1}$ were introduced in (1.2.6). Let Q be a cube in $\mathbb{R}^d$ centered at some point $c_Q \in \mathbb{R}^d$ and $f \in L^2_+(Q)$ be a density such that $\int_Q f =: \theta_Q$. Then, we will denote by ρ_Q the kernel $\theta\rho_{\theta,\varepsilon}$ refocused at c_Q with a small abuse of notation (indeed, ρ_Q also depends on f):

$$\rho_Q(x) = \theta_Q \rho_{\theta_Q,\varepsilon}(x - c_Q).$$

The main result of this section is the following theorem

Theorem 3.2.2 (Local estimate)**.** *Let us set $Q_0 = (0, L)^d$ for some $L > 0$. There exists $C > 0$ only depending on α, ρ and d such that for all $f \in L^2_+(Q_0)$ with $\int_{Q_0} f = \theta$, we have*

- *If* $\operatorname{supp} \rho_{Q_0} \subset Q_0$ *then, there exists $u \in H_0^1(Q_0)$ such that $\nabla \cdot u = f - \rho_{Q_0}$ and*

  $$d_\varepsilon^\alpha(f, \rho_{Q_0}) \le M_\varepsilon^\alpha(u) \le C\big\{\theta^\alpha L + \varepsilon^{\gamma_2}\|f\|_{L^2}^2\big\} \quad \text{and} \quad \|u\|_{L^1} \le C\,L\,\theta.$$

- *Otherwise, there exists $u \in H_0^1(\widetilde{Q}_0)$ such that*

  $$d_\varepsilon^\alpha(f, \rho_{Q_0}) \le M_\varepsilon^\alpha(u) \le C\varepsilon^{\gamma_2}\|f\|_{L^2}^2 \quad \text{and} \quad \|u\|_{L^1} \le C\,L\,\theta,$$

  where $\widetilde{Q}_0 = 2\operatorname{supp}\rho_{Q_0} := B(c_{Q_0}, 2R_{\theta,\varepsilon})$.

Remark 3.2.3. The Dirichlet term, $\varepsilon^{\gamma_2}\|f\|_{L^2}^2$, in the estimates above is easily understandable. Indeed, if ε is very large so that one can get rid of the first term in the energy M_ε^α, then, one can use a classical Dirichlet type estimate, that is Theorem 3.2.4 below. On the contrary, for ε very small, the Γ-limit result on M_ε^α tells us that these energies are close to M^α so that it is natural to hope a similar estimate as the one for M^α: that is to say an estimate from above by $\theta^\alpha L$ (see [12]).

The main difficulty to prove Theorem 3.2.2 is the non subadditivity of the pseudo-distances $\lambda_\varepsilon^\alpha$. Indeed, our proof is based on a dyadic construction used by Q. Xia in [64] to prove Proposition 3.2.1 (see also [12]). This gives a singular vector measure u which is concentrated on a graph. Since M_ε^α contains a term involving the L^2 norm of ∇u, we have to regularize u by taking a convolution with the kernel $\rho_{\theta,\varepsilon}$ on each branch of the graph (θ being the mass traveling on it). Unfortunately in this way, two different branches are no longer disjoints.

It is useful to see that we have a first candidate for the minimization problem (3.1.1). This candidate is of the form $u = \nabla\phi$, where ϕ is the solution of the Dirichlet problem

$$\begin{cases} \Delta\phi &= f^+ - f^- \quad \text{in} \quad Q, \\ \phi &= 0 \qquad\qquad \text{on} \quad \partial Q. \end{cases}$$

Then $u = \nabla\phi$ satisfies $\nabla\cdot u = f^+ - f^-$ in Q and $u(x) \in \mathbb{R}\,n$ a.e. on ∂Q where n stands for the external unit normal vector to ∂Q. Alternatively, one could consider Neumann homogeneous boundary conditions for ϕ rather than Dirichlet boundary conditions. Then, one would obtain $u(x)\cdot n = 0$ a.e. on ∂Q. Theorem 3.2.4 below gives a better result in the sense that the candidate u vanishes at the boundary:

Theorem 3.2.4. *Let $Q_0 = (0,L)^d$ be a cube of side length $L > 0$. There exists $C > 0$ only depending on d such that for all $f \in L_0^2(Q_0)$, there exists $u \in H_0^1(Q_0,\mathbb{R}^2)$ solving $\nabla\cdot u = f$ and satisfying $\|u\|_{L^1(Q_0)} \le CL\,\|f\|_{L^1(Q_0)}$ together with*

$$\|u\|_{H_0^1(Q_0)} := \left(\int_{Q_0} |\nabla u|^2\right)^{1/2} \le C\,\|f\|_{L^2(Q_0)},$$

where $L_0^2(Q_0) = \left\{f \in L^2(Q_0) \ : \ \int_{Q_0} f(x)\,\mathrm{d}x = 0\right\}$.

For a proof of this theorem, see, for instance, Theorem 2 in [17]: the only difference with Theorem 3.2.4 is that we add the estimate $\|u\|_{L^1(Q_0)} \le CL\,\|f\|_{L^1(Q_0)}$ which can be easily obtained following the proof of J. Bourgain and H. Brezis. The corresponding property formulated on a Lipschitz bounded connected domain Ω is also true (see Theorem 2' in [17]) except that the constant C could depend on Ω in this case.

Of course, this candidate is usually not optimal for (3.1.1) and this does not allow for a good estimate because of the first term in the definition of M_ε^α. For this reason, we have to use the dyadic construction of Q. Xia up to a certain level ("diffusion level") from which we simply use Theorem 3.2.4.

3.2.1 Dyadic decomposition and "diffusion level" of the source term

Let us call "dyadic descent" of $Q_0 = (0,L)^d$ the set $\mathcal{Q} = \bigcup_{j\geq 0} \mathcal{Q}_j$, where $\mathcal{Q}_j$ is the j^{th} "dyadic generation":

$$\mathcal{Q}_j = \left\{(x_1,\dots,x_d) + 2^{-j}Q_0 \ : \ x_i \in \{k2^{-j}L \ : \ 0 \leq k \leq 2^j - 1\} \quad \text{for} \quad i = 1,\dots,d\right\}.$$

Note that $\mathrm{Card}(\mathcal{Q}_j) = 2^{jd}$. For each $Q \in \mathcal{Q}$, let us define

- $\mathcal{D}(Q)$: the descent of Q, the family of all dyadic cubes contained in Q.
- $\mathcal{A}(Q)$: the ancestry of Q, the family of all dyadic cubes containing Q.
- $\mathcal{C}(Q)$: the family of children of Q composed of the 2^d biggest dyadic cubes strictly included in Q.
- $F(Q)$: the father of Q, the smallest dyadic cube strictly containing Q.

We now remind the dyadic construction described in [64] which irrigates f from a point source. We first introduce some notations: fix a function $f \in L^2_+(Q_0)$ with integral θ and let $Q \in \mathcal{Q}$ be a dyadic cube centered at $c_Q \in \mathbb{R}^d$. Then we introduce θ_Q the mass associated to the cube Q as

$$\theta_Q = \int_Q f \quad .$$

If $\theta_Q \neq 0$, we also define the kernel associated to Q through

$$\overline{\rho}_Q(x) = \rho_R(x), \tag{3.2.1}$$

where ρ_R is defined in (1.2.4) for

$$R = R_Q := \varepsilon^\gamma \theta_Q^{\frac{1-\gamma}{d-1}} \quad , \quad \gamma = \frac{\gamma_2}{d+1} \quad .$$

Here γ was defined in Define also the weighted recentered kernel by

$$\rho_Q(x) = \theta_Q \overline{\rho}_Q(x - c_Q) \tag{3.2.2}$$

if $\theta_Q \neq 0$ and $\rho_Q(x) = 0$ otherwise. Lastly, we introduce the point source associated to the cube Q as

$$\mathcal{S}_Q := \theta_Q \times \text{Dirac measure at point } c_Q.$$

We are now able to construct a vector measure X such that $M^\alpha(X) < +\infty$. First define the measures X_Q as below:

$$X_Q = \sum_{Q' \in \mathcal{C}(Q)} \theta_{Q'} \, n_{Q'} \, \mathcal{H}^1_{|[c_Q, c_{Q'}]}, \tag{3.2.3}$$

where $n_{Q'} = \dfrac{c_{Q'} - c_Q}{\|c_{Q'} - c_Q\|}$. Then, we have

$$\nabla \cdot X_Q = \sum_{Q' \in \mathcal{C}(Q)} \mathcal{S}_{Q'} - \mathcal{S}_Q$$

and the energy estimate

$$M^\alpha(X_Q) \leq 2^{d-2}\theta_Q^\alpha \operatorname{diam}(Q),$$

where $\operatorname{diam}(Q)$ stands for the diameter of Q. Finally, the measure $X = \sum_{Q\in\mathcal{Q}} X_Q$ solves $\nabla \cdot X = f - \mathcal{S}_{Q_0}$ and satisfies

$$M^\alpha(X) \leq C\theta^\alpha \operatorname{diam}(Q_0).$$

Indeed, it is enough to apply the following lemma with $\lambda = \alpha$:

Lemma 3.2.5. *Let $Q \in \mathcal{Q}$ and $\lambda \in]1-1/d, 1]$. There exists a constant $C = C(\lambda, d)$ such that*

$$\sum_{Q'\in\mathcal{D}(Q)} \theta_{Q'}^\lambda \operatorname{diam}(Q') \leq C\theta_Q^\lambda \operatorname{diam}(Q).$$

Proof. Let $j_0 \geq 0$ be such that $Q \in \mathcal{Q}_{j_0}$. The definition of $\mathcal{D}(Q)$, the Jensen inequality and the fact that $d - 1 - \lambda d < 0$ give

$$\begin{aligned}
\sum_{Q'\in\mathcal{D}(Q)} \theta_{Q'}^\lambda \operatorname{diam}(Q') &= \sum_{j\geq 0} 2^{-j}\operatorname{diam}(Q) \sum_{Q'\in\mathcal{D}(Q)\cap\mathcal{Q}_{j_0+j}} \theta_{Q'}^\lambda \\
&\leq \operatorname{diam}(Q)\sum_{j\geq 0} 2^{-j}2^{jd}\left(2^{-jd}\sum_{Q'\in\mathcal{D}(Q)\cap\mathcal{Q}_{j_0+j}} \theta_{Q'}\right)^\lambda \\
&\leq \theta_Q^\lambda \operatorname{diam}(Q)\sum_{j\geq 0} 2^{j(d-1-\lambda d)} \\
&\leq C\theta_Q^\lambda \operatorname{diam}(Q).
\end{aligned}$$

□

Now, the idea is to replace each term in (3.2.3) by its convolution with the kernel $\overline{\rho}_{Q'}$. Unfortunately, this will make appear extra divergence terms around each node. We have to modify X so as to make this extra divergence vanish using, for instance, Theorem 3.2.4. Furthermore, we cannot follow the construction for all generations $j \geq 1$, otherwise the "enlarged edges" (convolution of a dyadic edge and the kernel $\rho_{\theta,\varepsilon}$) may overlap. This is the reason why we introduce the following definition:

Definition 3.2.6 ("Diffusion level")**.** For a cube Q_0 and $f \in L_+^2(Q_0)$ we define the set $\mathcal{D}(Q_0, f)$ or $\mathcal{D}(f) \subset \mathcal{Q}$ as the maximal element for the inclusion in the set

$$\Lambda = \{D \subset \mathcal{Q} \;:\; \forall Q \in D,\ \mathcal{A}(Q) \cup \mathcal{C}(F(Q)) \subset D \quad \text{and} \quad \operatorname{supp}\rho_Q \subset Q\}.$$

If $\Lambda = \emptyset$, that is $\operatorname{supp}\rho_{Q_0} \not\subseteq Q_0$, we take the convention $\mathcal{D}(f) = \emptyset$. For all $x \in Q_0$, define also the "generation index" of x associated to f as

$$j(f,x) = \max\{j \;:\; \exists Q \in \mathcal{D}(f) \cap \mathcal{Q}_j,\ x \in Q\} \in \mathbb{N} \cup \{\pm\infty\},$$

where the convention $\max(\emptyset) = -\infty$ has been used.

In this way, each cube in $\mathcal{D}(f)$ contains the support of its associated kernel. Moreover, if Q is an element of $\mathcal{D}(f)$, then all its ancestry and its brothers (i.e. elements of the set $\mathcal{C}(F(Q)))$ are elements of $\mathcal{D}(f)$. $\mathcal{D}(f)$ can be constructed by induction as follows: first take $j=0$ and $\mathcal{D}(f)=\emptyset$. If $\operatorname{supp}\rho_{Q_0}\subset Q_0$ then add Q_0 to the set $\mathcal{D}(f)$ and j is replaced by $j+1$. For all cubes Q in $\Lambda\cap\mathcal{Q}_{j-1}$: if all cubes $Q'\in\mathcal{C}(Q)\subset\mathcal{Q}_j$ are such that their associated kernels are supported on Q' then $\mathcal{D}(f)$ is replaced by $\mathcal{D}(f)\cup\mathcal{C}(Q)$. If $\mathcal{D}(f)$ has been changed at this stage j is replaced by $j+1$ and the preceding step is reiterated. This process is repeated for $j\geq 1$ until it fails.

Let $\mathcal{D}_{min}(f)$ be the set of all cubes in $\mathcal{D}(f)$ which are minimal for the inclusion. If $\mathcal{D}_{min}(f)\neq\emptyset$, we also define

$$D(f)=\bigcup_{Q\in\mathcal{D}_{min}(f)} Q \quad .$$

Note that this is actually a disjoint union: two distinct cubes in $\mathcal{D}_{min}(f)$ are disjoint. Indeed, for Q, $Q'\in\mathcal{D}_{min}(f)\subset\mathcal{Q}$, either $Q\cap Q'=\emptyset$ or Q and Q' are comparable: $Q\subset Q'$ or $Q'\subset Q$. In the last case, since Q and Q' are minimal, we deduce that $Q=Q'$.

Moreover, it is not difficult to see that, if $\mathcal{D}_{min}(f)\neq\emptyset$, then $D(f)=\{x\in Q_0 \ : \ j(f,x) \text{ is finite}\}$ and also that $\overline{f}(x)=0$ whenever $j(f,x)=+\infty$, where $\overline{f}$ is the precise representative of f (i.e. the limit of the mean values of f on small cubes). Indeed, assume that $Q\in\mathcal{D}(f)$ is a cube of side length L_Q. Then, by definition, $\operatorname{supp}\rho_Q\subset Q$ and for some constant C depending on ρ and for $\nu=\frac{1-\gamma}{d-1}$, one has $\varepsilon^\gamma\theta_Q^\nu\leq CL_Q$ and so

$$⨍_Q f := L_Q^{-d}\theta_Q\leq\varepsilon^{-\gamma/\nu}L_Q^{1/\nu-d}.$$

Since $1/\nu-d=\frac{(d-1)(\alpha d-d+1)}{d+1}$ is positive, we deduce that L_Q cannot be arbitrarily small if there exists $x\in Q$ such that $\overline{f}(x)>0$. Moreover, if $f(x)\geq\eta$ a.e. for some $\eta>0$, then there exists some constant $C_\eta>0$ depending on η, ε, d and α such that

$$\forall Q\in\mathcal{D}(f),\ L_Q\geq C_\eta. \tag{3.2.4}$$

In particular, one can deduce that $\mathcal{D}_{min}(f)=\emptyset$ if and only if $\mathcal{D}(f)=\emptyset$ or $f(x)=0$ a.e. Indeed, if $\mathcal{D}(f)=\emptyset$, then it is clear that $\mathcal{D}_{min}(f)=\emptyset$. Conversely, assume that $\mathcal{D}_{min}(f)\neq\emptyset$ (i.e. $Q_0\in\mathcal{D}(f)$) and that there exists $x\in Q_0$ such that $\overline{f}(x)>0$. Since $\bigcup_{Q\in\mathcal{D}(f)}\partial Q$ is negligible for the Lebesgue measure, one can assume that $x\in\bigcup_{Q\in\mathcal{D}(f)}\mathring{Q}$. Then $0\leq j(f,x)<+\infty$ and so there exists a minimal cube $Q\in\mathcal{D}(f)$ containing x. Then $Q\in\mathcal{D}_{min}(f)$. Indeed, if $Q'\in\mathcal{D}(f)$ and $Q'\subsetneq Q$, then $\mathcal{A}(Q)\subset\mathcal{D}(f)$ and there exists $Q''\in\mathcal{A}(Q)$ such that $Q''\subsetneq Q$ and $x\in Q''$ which is a contradiction.

We are now able to define two approximations of f which are useful for our problem. The first is a dyadic approximation of f by an atomic measure,

$$\Lambda_\varepsilon f=\begin{cases}\displaystyle\sum_{Q\in\mathcal{D}_{min}(f)}\mathcal{S}_Q & \text{if } \mathcal{D}_{min}(f)\neq\emptyset,\\ \mathcal{S}_{Q_0} & \text{otherwise,}\end{cases}$$

where we recall the definition of $\mathcal{S}_Q:=\theta_Q\delta_{c_Q}$. We also define an approximation in $H^1(Q_0)$,

$$\lambda_\varepsilon f=\begin{cases}\displaystyle\sum_{Q\in\mathcal{D}_{min}(f)}\rho_Q & \text{if } \mathcal{D}_{min}(f)\neq\emptyset,\\ \rho_{Q_0} & \text{otherwise,}\end{cases}$$

where ρ_Q is defined in (3.2.2). The following result shows in which sense $\lambda_\varepsilon f$ is an approximation of f and justifies the term "diffusion level". Indeed, this proposition indicates that we get a good estimate by using a local diffusion from $\lambda_\varepsilon f$ to f, i.e. minimizing $\int_Q |\nabla u|^2$ over the constraint $\nabla \cdot u = \lambda_\varepsilon f - f$ for all $Q \in \mathcal{D}_{min}(f)$ (see Theorem 3.2.4).

Proposition 3.2.7. *There exists a constant $C > 0$ depending on d and ρ such that for all $f \in L^2_+(Q_0)$,*

$$d^\alpha_\varepsilon(\lambda_\varepsilon f, f) + d^\alpha(\Lambda_\varepsilon f, f) \leq C\, \varepsilon^{\gamma_2} \|f\|^2_{L^2(Q_0)}.$$

More precisely, if $\operatorname{supp} \rho_{Q_0} \subset Q_0$*, there exists $u \in H^1_0(Q_0)$ such that $\nabla \cdot u = f - \lambda_\varepsilon f$ as well as*

$$M^\alpha_\varepsilon(u) \leq C\, \varepsilon^{\gamma_2} \|f\|^2_{L^2} \quad \textit{and} \quad \|u\|_{L^1} \leq C \operatorname{diam}(Q_0) \|f\|_{L^1}.$$

If $\operatorname{supp} \rho_{Q_0} \not\subseteq Q_0$ *the same estimates hold but the condition $u \in H^1_0(Q_0)$ has to be replaced by $u \in H^1_0(\widetilde{Q}_0)$, where $\widetilde{Q}_0$ is a cube containing Q_0 and* $\operatorname{supp} \rho_{Q_0}$.

Proof. First assume that $\operatorname{supp} \rho_{Q_0} \subset Q_0$ i.e. $Q_0 \in \mathcal{D}(f)$. If $\mathcal{D}_{min}(f) = \emptyset$, then $f(x) = 0$ a.e. and the proposition is trivial. Hence, one can assume that $\mathcal{D}_{min}(f) \neq \emptyset$. Then f is supported on $D(f)$ and $\mathcal{D}_{min}(f) =: \{Q_i\}_{i \in I}$ is a finite or countable partition of $D(f)$ (up to sets of measure 0, corresponding to some boundaries of cubes).

Denote for simplicity $D_i := \operatorname{diam}(Q_i)$, $f_i := f \mathbf{1}_{Q_i}$ (restriction of f to Q_i), $\theta_i := \theta_{Q_i}$ and $\rho_i := \rho_{Q_i} = \theta_i\, \rho_{R_i}$ for $i \in I$, where

$$R_i := R_{Q_i} = \varepsilon^\gamma \theta_i^{\frac{1-\gamma}{d-1}}.$$

Since Q_i is minimal in $\mathcal{D}(f)$, we deduce that, for some constants $C, C' > 0$,

$$C' R_i \leq D_i \leq C R_i. \tag{3.2.5}$$

Indeed, the first inequality follows from the fact that $\operatorname{supp} \rho_i \subset Q_i$ and $\operatorname{diam}(\operatorname{supp} \rho_i) = c R_i$ for some constant c depending on ρ. For the second inequality observe that, since Q_i is minimal, there exists $Q \in \mathcal{C}(Q_i)$ such that $\operatorname{supp} \rho_Q \not\subseteq Q$ and hence $R_Q \geq c' \operatorname{diam}(Q) = c'/2 D_i$ for some constant $c' > 0$ depending on ρ. Since $\theta_Q \leq \theta_{Q_i} = \theta_i$, one has $R_Q \leq R_i$ and the second inequality follows.

Now, Theorem 3.2.4 allows us to find $u_i \in H^1_0(Q_i)$ such that $\nabla \cdot u_i = g_i$, $\|u_i\|_{H^1(Q_i)} \leq C\, \|g_i\|_{L^2(Q_i)}$ and $\|u_i\|_{L^1(Q_i)} \leq C\, D_i \|g_i\|_{L^1(Q_i)}$, where $g_i := f_i - \rho_i$. Since u_i vanishes at ∂Q_i, one can extend u_i by 0 out of Q_i to get a function in $H^1(\mathbb{R}^d)$: for the sake of simplicity, this function is still denoted by u_i. Consequently, $u = \sum_i u_i$ belongs to $H^1_0(Q_0)$ and $\nabla \cdot u = f - \lambda_\varepsilon f$. It remains to estimate $M^\alpha_\varepsilon(u)$ and $\|u\|_{L^1(Q_0)}$. First of all,

$$\|u\|_{L^1(Q_0)} \leq \sum_i \|u_i\|_{L^1(Q_i)} \leq C \operatorname{diam}(Q_0) \sum_i \|g_i\|_{L^1(Q_i)}$$

and the inequality $\|g_i\|_{L^1(Q_i)} \leq 2\theta_i$ leads to $\|u\|_{L^1} \leq 2C \operatorname{diam}(Q_0) \|f\|_{L^1}$ as required.

Let us compute the L^2-norm of ρ_i:

$$\|\rho_i\|^2_{L^2(Q_i)} = \theta_i^2\, \|\rho_{R_i}\|^2_{L^2(Q_i)} = \theta_i^2 R_i^{-d} \|\rho\|^2_{L^2(Q_i)} = C \theta_i^2 R_i^{-d}.$$

By a Cauchy-Schwarz inequality,

$$\theta_i^2 = \left(\int_{Q_i} f_i\right)^2 \leq |Q_i| \|f_i\|^2_{L^2(Q_i)} = D_i^d \|f_i\|^2_{L^2(Q_i)} \tag{3.2.6}$$

which, together with (3.2.5), gives

$$\|\rho_i\|^2_{L^2(Q_i)} \leq C\, R_i^d \|f_i\|^2_{L^2(Q_i)} R_i^{-d} = C\, \|f_i\|^2_{L^2(Q_i)}.$$

Since $\|u_i\|_{H^1(Q_i)} \leq C\|f_i - \rho_i\|_{L^2(Q_i)}$, we get $\|u_i\|_{H^1(Q_i)} \leq C\|f_i\|_{L^2(Q_i)}$. Now, because the energy M_ε^α is local and since each u_i is supported on Q_i, one has

$$M_\varepsilon^\alpha(u) = \sum_{i=1}^n M_\varepsilon^\alpha(u_i) = \sum_{i=1}^n \left(\varepsilon^{-\gamma_1} \int_{Q_i} |u_i|^\beta + \varepsilon^{\gamma_2} \int_{Q_i} |\nabla u_i|^2\right).$$

By construction of u_i, one has

$$\int_{Q_i} |\nabla u_i|^2 \leq \|u_i\|^2_{H^1(Q_i)} \leq C\|g_i\|^2_{L^2(Q_i)} \leq 2C\left(\|\rho_i\|^2_{L^2(Q_i)} + \|f_i\|^2_{L^2(Q_i)}\right) \leq C'\|f_i\|^2_{L^2(Q_i)}.$$

It remains to estimate the first term. First of all, we use the Hölder and Poincaré inequalities as follows:

$$\int_{Q_i} |u_i|^\beta \leq |Q_i|^{1-\beta/2} \left(\int_{Q_i} |u_i|^2\right)^{\beta/2} \leq D_i^{d-d\beta/2} \left(D_i^2 \int_{Q_i} |\nabla u_i|^2\right)^{\beta/2} \leq D_i^\nu \|f_i\|^\beta_{L^2(Q_i)},$$

where $\nu = \beta + d - \frac{d\beta}{2}$. In view of (3.2.6) and (3.2.5), we have

$$D_i \leq CR_i = C\varepsilon^\gamma \theta_i^{\frac{1-\gamma}{d-1}} \leq C\varepsilon^\gamma (D_i^{\frac{d}{2}} \|f_i\|_{L^2(Q_i)})^{\frac{1-\gamma}{d-1}}$$

and, introducing $\delta := 1 - \frac{d(1-\gamma)}{2(d-1)}$,

$$D_i^\delta \leq C\varepsilon^\gamma \|f_i\|_{L^2}^{\frac{1-\gamma}{d-1}}. \tag{3.2.7}$$

Finally, since $-\gamma_1 + \frac{\gamma\nu}{\delta} = \gamma_2$ and $\beta + \frac{\nu(1-\gamma)}{\delta(d-1)} = 2$, we get

$$\varepsilon^{-\gamma_1} \int_{Q_i} |u_i|^\beta \leq C\varepsilon^{-\gamma_1 + \frac{\gamma\nu}{\delta}} \|f_i\|_{L^2(Q_i)}^{\beta + \frac{\nu(1-\gamma)}{\delta(d-1)}} = C\varepsilon^{\gamma_2} \|f_i\|^2_{L^2(Q_i)}.$$

The proof of the second inequality is quite similar but easier:

$$d^\alpha(\Lambda_\varepsilon f, f) \leq \sum_{i=1}^n d^\alpha(\mathcal{S}_{Q_i}, f_i) \leq \sum_{i=1}^n \theta_i^\alpha D_i \quad .$$

Once again, applying (3.2.6) and then (3.2.7), we get

$$d^\alpha(\Lambda_\varepsilon f, f) \leq C\, \varepsilon^{\gamma_2} \|f\|^2_{L^2}.$$

In the case where $\operatorname{supp} \rho_{Q_0} \not\subseteq Q_0$, i.e. $R_{Q_0} := \varepsilon^\gamma \theta_Q^{\frac{1-\gamma}{d-1}} \geq CL$ (L being the side length of Q_0 and C a constant depending on ρ), the proof is the same. Indeed, we just apply Theorem 3.2.4 to $g = f - \rho_{Q_0}$ and the same computations as above lead to the same result. □

3.2.2 Proof of the local estimate, Theorem 3.2.2

Let $Q_0 = (0,L)^d$, $L > 0$ and $f \in L^2_+(Q_0)$ with $\int_{Q_0} f = \theta$. In the case where $\operatorname{supp}\rho_{Q_0} \not\subset Q_0$, Theorem 3.2.2 is a particular case of Proposition 3.2.7. Consequently, one can assume that $\operatorname{supp}\rho_{Q_0} \subset Q_0$ i.e. $Q_0 \in \mathcal{D}(f)$. In the case where $\mathcal{D}(f) = \{Q_0\}$, one has $\lambda_\varepsilon f = \rho_{Q_0}$ and Theorem 3.2.2 is a consequence of Proposition 3.2.7 as well. For this reason, one can assume that $\mathcal{C}(Q_0) \subset \mathcal{D}(f)$. Moreover, up to replacing f by $f + \eta$ for some small constant $\eta > 0$ and passing to the limit when $\eta \to 0$, one can assume that $\mathcal{D}(f)$ is finite. Indeed, in view of (3.2.4), $\mathcal{D}(f+\eta)$ is finite since for all $Q \in \mathcal{D}(f+\eta)$, $\operatorname{diam}(Q) \geq C_\eta > 0$.

Our aim is to prove that there exists $C > 0$ only depending on α, d and ρ such that

$$d^\alpha_\varepsilon(f,\rho_{Q_0}) \leq C\big\{\theta^\alpha L + \varepsilon^{\gamma_2}\|f\|^2_{L^2(Q_0)}\big\}.$$

The idea of the proof is to approximate the vector field $X = \sum X_Q$ of the previous section (see (3.2.3)) by a vector field in H^1 using the kernel ρ. In this part, we will use the notations of the previous section: in particular, the definition of $\mathcal{D}(f)$ in Definition 3.2.6, the measures X_Q in (3.2.3) and $X = \sum_{Q\in\mathcal{D}(f)} X_Q$.

Since $\mathcal{C}(Q_0) \subset \mathcal{D}(f)$, we can construct the regularized vector field Y by the formula

$$Y = \sum_{\substack{Q\in\mathcal{D}(f)\\ Q\neq Q_0}} Z_Q,$$

where, for all $Q \in \mathcal{D}(f)$ such that $Q \neq Q_0$ (see Figure 3.1),

$$Z_Q := \theta_Q\, n_Q\, \overline{\rho}_Q * \mathcal{H}^1_{|[c_{F(Q)},c_Q]}, \tag{3.2.8}$$

n_Q being the normalized vector $n_Q = \frac{c_Q - c_{F(Q)}}{\|c_Q - c_{F(Q)}\|}$ and $\overline{\rho}_Q$ being defined in (3.2.1).

By definition of the kernel $\overline{\rho}_Q$, one has

$$M^\alpha_\varepsilon(Z_Q) \leq C\theta^\alpha_Q \operatorname{diam}(Q). \tag{3.2.9}$$

This a consequence of the choice of R_Q as a minimizer in (1.2.5). Indeed, for the sake of simplicity, let us assume that $\operatorname{supp}\rho$ is the unit ball centered at the origin. Then Z_Q is concentrated on a strip of width R_Q around the segment $S = [c_{F(Q)}, c_Q]$, i.e.

$$\operatorname{supp} Z_Q \subset \tilde{S} := \{x \in \mathbb{R}^d \ :\ \operatorname{dist}(x,S) \leq R_Q\} \tag{3.2.10}$$

and Z_Q satisfies the two estimates

$$\|Z_Q\|_{L^\infty} \leq C\theta_Q R_Q^{1-d} \quad \text{and} \quad \|\nabla Z_Q\|_{L^\infty} \leq C\theta_Q R_Q^{-d}. \tag{3.2.11}$$

Then, the same computations as in (1.2.5) and the fact that $R_Q \leq \operatorname{diam}(Q)$ give (3.2.9).

Let us estimate the L^1 norm of Y which has to be controlled by θ as stated in Theorem 3.2.2:

$$\|Y\|_{L^1(Q_0)} \leq \sum_{j\geq 1}\ \sum_{Q\in\mathcal{D}(f)\cap\mathcal{Q}_j} \|Z_Q\|_{L^1(\tilde{S})} \leq \sum_{j\geq 1}\ \sum_{Q\in\mathcal{D}(f)\cap\mathcal{Q}_j} \theta_Q\, L\, 2^{-j} \ = L\,\theta.$$

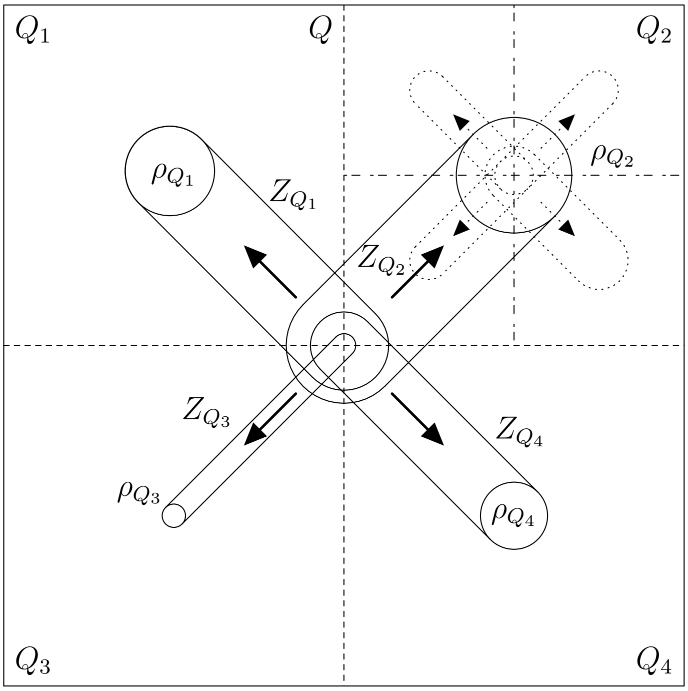

Figure 3.1 – A square Q and its 4 dyadic children Q_i with the associated vector field Z_Q

Note that

$$\nabla \cdot Y = \rho_{Q_0} - h - \lambda_\varepsilon f,$$

where h stands for the extra divergence. h can be written as

$$h = \sum_{Q \in \mathcal{D}_{fr.}} \left\{ \rho_Q - \sum_{Q' \in \mathcal{C}(Q)} \rho_{Q',Q} \right\},$$

where $\rho_{Q',Q}$ represents the kernel $\rho_{Q'}$ translated at c_Q, center of Q, and, for the sake of simplicity, the set of all cubes Q such that $\mathcal{C}(Q) \subset \mathcal{D}(f)$ has been denoted by $\mathcal{D}_{fr.}$:

$$\mathcal{D}_{fr.} := \{Q \in \mathcal{D}(f) \ : \ \mathcal{C}(Q) \subset \mathcal{D}(f)\}.$$

Since $\nabla \cdot Y = \rho_{Q_0} - f + (f - \lambda_\varepsilon f) - h \neq \rho_{Q_0} - f$, we have to slightly modify the vector field Y. This will be done replacing Y by

$$V = Y + V_1 + V_2,$$

where V_1, $V_2 \in H^1(Q_0, \mathbb{R}^d)$ are constructed so that $\nabla \cdot V_1 = h$ and $\nabla \cdot V_2 = \lambda_\varepsilon f - f$. The construction of V_1 and the estimate of $M_\varepsilon^\alpha(V_1)$, $\|V_1\|_{L^1}$ will be the object of the first step. In the second step we prove that $M_\varepsilon^\alpha(Y) \leq C\theta^\alpha L$. Then, Proposition 3.2.7 allows us to construct $V_2 \in H^1$ such that $\nabla \cdot V_2 = \lambda_\varepsilon f - f$ with an estimate on $M_\varepsilon^\alpha(V_2)$ and $\|V_2\|_{L^1}$.

First step: Correction at the nodes, construction of V_1. For all $Q \in \mathcal{D}_{fr.}$, let B_Q be the support of ρ_Q. Since $\operatorname{supp}\rho$ has been supposed to be the unit ball centered at the origin and $\rho_Q(x) = \theta_Q \rho_{R_Q}(x - c_Q)$, we have $B_Q = B(c_Q, R_Q) \subset Q$. Let us define the extra divergence corresponding to this node,

$$h_Q = \rho_Q - \sum_{Q' \in \mathcal{C}(Q)} \rho_{Q',Q} \quad .$$

For each $Q \in \mathcal{D}_{fr.}$, thanks to Theorem 3.2.4, we can find $V_Q \in H^1_0(B_Q)$ such that $\nabla \cdot V_Q = h_Q$ and $\|V_Q\|_{H^1(B_Q)} \leq C \, \|h_Q\|_{L^2(B_Q)}$. But in this case, because h_Q is radial up to a translation, we essentialy use the proposition in dimension 1 which is quite easy and gives better estimates. Let us give more details on this point:

Lemma 3.2.8. *Let $d \geq 1$ and $B = B(0,R) \subset \mathbb{R}^d$ be a ball centered at the origin. There exists a constant $C > 0$ only depending on d such that the following holds:*

Let $F \in L^\infty(B)$ be a radial function: i.e. for a.e. $x \in B$, $F(x) = f(|x|)$ for some $f \in L^\infty(0,R)$. Assume that $\int_B F = 0$. Then, there exists a radial function $V \in W^{1,\infty}_0(B, \mathbb{R}^d)$ such that $\nabla \cdot V = F$ and

$$\|\nabla V\|_{L^\infty(Q_0)} \leq C \, \|F\|_{L^\infty(Q_0)} \quad .$$

Proof. First of all, by a scaling argument, one can assume that $R = 1$. The vector field $V : B \to \mathbb{R}^d$ defined by $V(x) = v(|x|)x$ for some Lipschitz function $v : \mathbb{R}^+ \to \mathbb{R}$ satisfies

$$\nabla \cdot V(x) = r^{1-d}[r^d v(r)]'$$

in the distributional sense. Thus, if v is chosen as

$$v(r) = r^{-d} \int_0^r f(s) s^{d-1} \, \mathrm{d}s,$$

then V solves the following problem:

$$\begin{cases} \nabla \cdot V(x) = F(x) & \text{on } B, \\ V(x) = 0 & \text{on } \partial B. \end{cases}$$

Moreover, for a.e. $x \in B$, we have $\nabla V(x) = v'(|x|)x \otimes \frac{x}{|x|} + v(|x|)\,\mathrm{Id}$, where Id is the matrix identity. In particular, we get $\|\nabla V\|_{L^\infty} \leq C(\|rv'(r)\|_{L^\infty} + \|v\|_{L^\infty})$. The second term in the RHS on the preceding equation is estimated by $\|v\|_{L^\infty} \leq r^{1-d}\|f\|_{L^\infty} r^{d-1} = \|f\|_{L^\infty}$. For the first term, one has $v'(r) = -dr^{-d-1}\int_0^r f(s)s^{d-1}\,\mathrm{d}s + r^{-1}f(r)$ and so $\|rv'(r)\|_{L^\infty} \leq C\|f\|_{L^\infty}$. Thus, $\|\nabla V\|_{L^\infty} \leq C\|F\|_{L^\infty}$. □

Applying Lemma 3.2.8 to $F = h_Q$ and $R = R_Q$ gives $V_Q \in H^1_0(B_Q)$ such that $\nabla \cdot V_Q = h_Q$ and

$$\|\nabla V_Q\|_{L^\infty(B_Q)} \leq C\theta_Q R_Q^{-d} \ , \quad \|\nabla V_Q\|_{L^1(B_Q)} \leq |B_Q| \|\nabla V_Q\|_{L^\infty(B_Q)} \leq C\theta_Q. \tag{3.2.12}$$

Moreover, since V_Q is supported on $B_Q = B(c_Q, R_Q)$, we deduce that $\|V_Q\|_{L^\infty(B_Q)} \leq R_Q\|\nabla V_Q\|_{L^\infty(B_Q)} \leq C\theta_Q R_Q^{1-d}$ so that V_Q satisfies the same estimate as (3.2.11). In particular, we get $M_\varepsilon^\alpha(V_Q) \leq C\theta_Q^\alpha \operatorname{diam}(Q)$. Now define

$$V_1 = \sum_{Q\in\mathcal{D}_{fr.}} V_Q.$$

Since $\|V_Q\|_{L^1(B_Q)} \leq CR_Q\|\nabla V_Q\|_{L^1(B_Q)} \leq C\operatorname{diam}(Q)\theta_Q$, Lemma 3.2.5 implies

$$\|V_1\|_{L^1(Q_0)} \leq C\operatorname{diam}(Q_0)\theta_{Q_0} \leq C'L\|f\|_{L^1(Q_0)}$$

as required. Then, using the definition of M_ε^α in (1.2.2) and the subadditivity of $x \to |x|^\beta$, one gets

$$M_\varepsilon^\alpha(V_1) \leq \varepsilon^{-\gamma_1} \sum_{Q\in\mathcal{D}_{fr.}} \int |V_Q|^\beta + 2\,\varepsilon^{\gamma_2} \int \sum_{Q,Q'\in\mathcal{D}_{fr.}\,:\,Q'\subset Q} |\nabla V_{Q'} : \nabla V_Q|\,, \tag{3.2.13}$$

where $A : B$ stands for the Euclidean product of two matrices $A = (A_{ij})_{1\leq i,j\leq d}$, $B = (B_{ij})_{1\leq i,j\leq d}$ of size $d\times d$: $A : B := \sum_{ij} A_{ij}B_{ij}$. For the estimate of $|\nabla V_1|^2$, we have used the identity $|\nabla V_1|^2 = \nabla V_1 : \nabla V_1 = \sum_{Q,Q'\in\mathcal{D}_{fath}} \nabla V_Q : \nabla V_{Q'}$. Since V_Q is supported on Q, $\nabla V_Q : \nabla V_{Q'}$ vanishes except when $Q\cap Q' \neq \emptyset$, i.e. $Q \subset Q'$ or $Q' \subset Q$, thus justifying the factor 2 and the inclusion $Q' \subset Q$ in (3.2.13).

We need to estimate the two terms in (3.2.13). Since $M_\varepsilon^\alpha(V_Q) \leq C\theta_Q^\alpha \operatorname{diam}(Q)$, thanks to Lemma 3.2.5, this term is less or equal than $C\theta^\alpha L$ as required. Using the inequality $\|fg\|_{L^1} \leq \|f\|_{L^\infty}\|g\|_{L^1}$, one can estimate the second term of (3.2.13) by

$$2\,\varepsilon^{\gamma_2} \sum_{Q,Q'\in\mathcal{D}_{fr.}\,:\,Q'\subset Q} \|\nabla V_Q\|_{L^\infty(B_Q)}\|\nabla V_{B_{Q'}}\|_{L^1(B_{Q'})}.$$

Note that it would be more natural to use a Cauchy-Schwarz inequality (L^2-L^2) at this step but, using it, we were not able to deduce the estimate by $\theta^\alpha L$. Once again, since $R_{Q'} \leq \operatorname{diam}(Q')$, we have

$$\|\nabla V_{Q'}\|_{L^1(B_{Q'})} \leq C\theta_{Q'} \leq \operatorname{diam}(Q')R_{Q'}^{-1}\theta_{Q'} = C\operatorname{diam}(Q')\varepsilon^{-\gamma}\theta_{Q'}^{1-\frac{1-\gamma}{d-1}}. \tag{3.2.14}$$

Since $1 - \frac{1}{d} < 1 - \frac{1-\gamma}{d-1} < 1$, Lemma 3.2.5 gives

$$\sum_{Q'\in\mathcal{D}_{fr.}\,:\,Q'\subset Q} \|\nabla V_{Q'}\|_{L^1(B_{Q'})} \leq C\varepsilon^{-\gamma}\operatorname{diam}(Q)\theta_Q^{1-\frac{1-\gamma}{d-1}}.$$

Now, elementary computations on exponents α, γ_2, γ and Lemma 3.2.5 give successively $\gamma_2 = (d+1)\gamma$, $\alpha = 2 - (d+1)\frac{1-\gamma}{d-1}$ and

$$C\varepsilon^{\gamma_2} \sum_{Q\in\mathcal{D}_{fr.}} \operatorname{diam}(Q)\theta_Q R_Q^{-d}\varepsilon^{-\gamma}\theta_Q^{1-\frac{1-\gamma}{d-1}} = C\sum_{Q\in\mathcal{D}_{fr.}} \operatorname{diam}(Q)\theta_Q^\alpha \leq C\theta^\alpha L.$$

Finally, we have obtained the desired inequality: $M_\varepsilon^\alpha(V_1) \leq C\,\theta^\alpha L$.

Second step: estimate of the energy of Y on the node set. In order to get estimates on Y, it is convenient to divide Q_0 into 2 domains: the node set N and its complementary N^c, where

$$N := \bigcup_{Q\in\mathcal{D}(f)} B(c_Q, cR_Q)$$

and $c > 0$ is a constant which will be chosen later. By analogy with V_1, one can write $Y_{|N}$ as a sum of vector fields Y_Q, where

$$Y_Q = \begin{cases} \mathbf{1}_{B(c_Q,cR_Q)}\left(Z_Q - \sum_{Q'\in\mathcal{C}(Q)} Z_{Q'}\right) & \text{if } Q \in \mathcal{D}_{fr.} \text{ (see (3.2.8))}, \\ \mathbf{1}_{B(c_Q,cR_Q)} Z_Q & \text{otherwise.} \end{cases}$$

Now, from (3.2.11), we deduce the estimates (3.2.12) satisfied by V_Q are also true for Y_Q and consequently, we obtain $M^\alpha_\varepsilon(Y, N) \leq C\, \theta^\alpha L$ as well (see (1.2.2) for the definition of $M^\alpha_\varepsilon(Y, N)$).

Third step: estimate of the energy of Y out of the node set. Reminding that

$$Y = \sum_{\substack{Q\in\mathcal{D}(f)\\ Q\neq Q_0}} Z_Q \quad ,$$

considering that M^α_ε is not subadditive (due to the term $|\nabla Y|^2$), the first thing to do is to understand to which extent the supports of Z_Q can intersect. To this aim, let us note that if the constant $c > 0$ in (3.2.2) is chosen equal to $\sqrt{d}$ or more, due to (3.2.10), then each Z_Q restricted to N^c is supported on Q (see figure 3.1): $\operatorname{supp} Z_Q \cap N^c \subset Q$. In particular, this implies that

$$\operatorname{supp} Z_Q \cap \operatorname{supp} Z_{Q'} \cap N^c \neq \emptyset \implies Q \cap Q' \neq \emptyset \implies Q \subset Q' \quad \text{or} \quad Q' \subset Q.$$

For this reason, $M^\alpha_\varepsilon(Y, N^c)$ can be estimated exactly in the same way as we did for the estimate of $M^\alpha_\varepsilon(V_1)$ in (3.2.13). Moreover, the Young inequality, $\|f * \mu\|_{L^1} \leq \|f\|_{L^1}|\mu|(\mathbb{R}^d)$, valid for all $f \in L^1(\mathbb{R}^d)$, $\mu \in \mathcal{M}(\mathbb{R}^d)$, and the definition of Z_Q in (3.2.8), easily give

$$\|\nabla Z_{Q'}\|_{L^1(Q')} \leq C\theta_{Q'} R^{-1}_{Q'} \operatorname{diam}(Q').$$

Since this estimate (which is the same as (3.2.14)) and (3.2.9) are the only ones we have used in the first step for the estimate of $M^\alpha_\varepsilon(V_1)$, we get $M^\alpha_\varepsilon(Y, N^c) \leq C\theta^\alpha L$ as well.

End of the proof of Theorem 3.2.2 Finally, the vector field $V = Y + V_1 + V_2$, where V_2 is given by Proposition 3.2.7, satisfies $\nabla \cdot V = \rho_{Q_0} - f$,

$$M^\alpha_\varepsilon(V) \leq 3\{M^\alpha_\varepsilon(Y) + M^\alpha_\varepsilon(V_1) + M^\alpha_\varepsilon(V_2)\} \leq C\{\theta^\alpha L + \varepsilon^{\gamma_2}\|f\|^2_{L^2}\}$$

and

$$\|V\|_{L^1} \leq \|Y\|_{L^1} + \|V_1\|_{L^1} + \|V_2\|_{L^1} \leq CL\|f\|_{L^1}.$$

3.3 Estimates between d_ε^α and the Wasserstein distance

Our aim is to prove an estimate on the pseudo-distances d_ε^α similar to Proposition 1.1.10. Because of the Dirichlet term in the definition of M_ε^α, d_ε^α cannot be estimated only by the Wasserstein distance W_1 but one has to add a term involving $\|f^+ - f^-\|_{L^2}$. Using Theorem 3.2.2, we are going to prove the following theorem:

Theorem 3.3.1. *Let $Q = (0, L)^d$ be a a cube of side length $L > 0$ in $\mathbb{R}^d$ and $\varepsilon \in (0, 1)$. There exists $C > 0$ only depending on α, d and L such that for all $f^+, f^- \in L^2_+(Q)$ with $\int_Q f^+ = \int_Q f^- = 1$, there exists $u \in H^1(\mathbb{R}^d)$ compactly supported on the set $Q_\varepsilon := \{x \in \mathbb{R}^d \ : \ \mathrm{dist}(x, Q) \leq C\varepsilon^\gamma\}$ satisfying $\nabla \cdot u = f := f^+ - f^-$ as well as*

$$d_\varepsilon^\alpha(f^+, f^-) \leq M_\varepsilon^\alpha(u) \leq C\, H\big(W_1^{1-d(1-\alpha)}(f^+, f^-) + \varepsilon^{\gamma_2}\|f\|^2_{L^2}\big) \quad \text{and} \quad \|u\|_{L^1} \leq C, \tag{3.3.1}$$

where $H : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is the scalar function defined by $H(x) = x + x^\lambda$ for some $\lambda \in (0, 1)$ depending on α, and W_1 stands for the Wasserstein distance associated to the Monge cost $(x, y) \to |x - y|$.

Remark 3.3.2. One can replace the condition $\int f^\pm = 1$ by $\int f^\pm = \theta \geq 0$. Then, the constant C will also depend on θ: $C = C(\theta, \alpha, d, L)$. However, we can easily check that C is locally bounded with respect to θ, i.e. it is uniform for bounded values of θ.

Remark 3.3.3. It is tempting to think that estimate (3.3.1) also holds when $H(x) = x$ which would be the natural choice. Indeed, if ε is taken very small, since M_ε^α Γ-converge to M^α and because of Proposition 1.1.10, one can expect that $d_\varepsilon^\alpha(f^+, f^-) \simeq d^\alpha(f^+, f^-) \leq CW_1(f^+, f^-)^{1-d(1-\alpha)}$. On the contrary, when ε is very large, because of Theorem 3.2.4, one can expect that $d_\varepsilon^\alpha(f^+, f^-) \simeq \varepsilon^{\gamma_2}\|f\|^2_{L^2}$. However, for technical reasons, due to the lack of subadditivity of the second term (Dirichlet energy) in the definition of M_ε^α, we were not able to reach the case $H(x) = x$.

Proof. Our method to prove this proposition is an adaptation of that of J.-M. Morel and F. Santambrogio in [51] (see also Proposition 6.16. page 64 in [12]).

Up to replacing (f^+, f^-) by $(f^+ - f^+ \wedge f^-, f^- - f^+ \wedge f^-)$, one can assume that $f^+ \wedge f^- = 0$, where for all $x \in Q$, $(f^- \wedge f^+)(x) = \inf(f^-(x), f^+(x))$. Indeed, it is sufficient to note that, if $\mu^\pm$ are two measures with the same mass and ν is a positive measure on Q then we have $W_1(\mu^+ + \nu, \mu^- + \nu) = W_1(\mu^+, \mu^-)$.

For the sake of simplicity, in all the proof, $C > 0$ will denote some constant only depending on α, d and L and big enough so that all the inequalities below are satisfied.

Let $f^+, f^- \in L^2_+(Q)$ be two densities on the cube $Q = (0, L)^d$ such that $\int_Q f^\pm = 1$. Chose an optimal transport plan Π between f^+ and f^- for the Monge-Kantorovich problem associated to the cost $c(x, y) = |x - y|$. Hence Π satisfies the constraint $P^\pm_\# \Pi = f^\pm(x)\,\mathrm{d}x$ where P^+ (resp. P^-) is the projection on the first variable x (resp. the second variable y) and $\mathrm{d}x$ is the Lebesgue measure. Moreover we have

$$\int_Q |x - y|\,\mathrm{d}\Pi(x, y) = W_1(f^+, f^-) =: W. \tag{3.3.2}$$

So as to use the local estimate of the previous part, let us classify the set of ordered pairs (x, y) with respect to the distance $|x-y|$. More precisely, for $j \geq 0$, set

$$X_j = \{(x,y) \in Q^2 \ : \ d_j \leq |x-y| < d_{j+1}\},$$

where $d_j = (2^j - 1)\, w$ and $w \in (0,1)$ will be chosen later. In particular, $d_0 = 0$ and X_j is empty if $d_j > \operatorname{diam}(Q)$, i.e. $j > J := \left\lfloor \ln_2 \left(\frac{\operatorname{diam}(Q)}{w} + 1 \right) \right\rfloor$. For this reason, one can restrict to integers $j \leq J \leq C(1 + |\ln w|)$: we will assume that $d_j \leq \operatorname{diam}(Q)$. Moreover, (3.3.2) immediately gives the estimate

$$\sum_j d_j \theta_j \leq W\,, \qquad \text{where} \qquad \theta_j = \Pi(X_j). \tag{3.3.3}$$

Next, for each integer $j \in [1, J]$, consider a uniform partition of Q into cubes Q_{jk}, $k = 1, \dots, K_j$, with side length d_{j+1}. It is easy to estimate K_j by

$$K_j \leq C d_{j+1}^{-d}. \tag{3.3.4}$$

For $j \geq 0$, set

$$\Pi_j = \Pi_{|X_j} \ ; \ \theta_j = \Pi(X_j); \ f_j^\pm = P_\#^\pm \Pi_j \quad \text{and} \quad f_j = f_j^+ - f_j^-,$$

Clearly, one has

$$\Pi = \sum_j \Pi_j \quad \text{and} \quad f^\pm = \sum_j f_j^\pm.$$

In the same way, for $j \geq 0$ and $1 \leq k \leq K_j$, set

$$\Pi_{jk} = \Pi_{|X_j \cap (Q_{jk} \times Q)} \ ; \ \theta_{jk} = \Pi_{jk}(Q^2) \quad \text{and} \quad f_{jk}^\pm = P_\#^\pm \, \Pi_{jk}$$

so that

$$\Pi_j = \sum_k \Pi_{jk}; \ \theta_j = \sum_k \theta_{jk} \quad \text{and} \quad f_j^\pm = \sum_k f_{jk}^\pm.$$

Π_{jk} represents the part of the transport plan Π corresponding to points in Q_{jk} which are sent at a distance comparable to d_{j+1}. In particular, f_{jk}^+ is supported on Q_{jk} and f_{jk}^- is supported on the cube $\widetilde{Q}_{jk}$ with the same center but twice the side length of Q_{jk}. As we did in (3.2.2), let us define ρ_{jk} the kernel associated to Q_{jk} by

$$\rho_{jk}(x) = \begin{cases} (R_{jk})^{-d} \rho(R_{jk}(x - c_{jk})) & \text{if } \theta_{jk} \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $\rho \in \mathcal{C}_c^1(\mathbb{R}^d, \mathbb{R}^+)$, $R_{jk} = \varepsilon^\gamma \theta_{jk}^{\frac{1-\gamma}{d-1}}$ and c_{jk} is the center of Q_{jk}. For the sake of simplicity, let us assume that $\operatorname{supp} \rho$ is the unit ball centered at the origin. Let $B_{jk} := B(c_{jk}, r_{jk})$ be the smallest ball containing $\widetilde{Q}_{jk}$ and $\operatorname{supp} \rho_{jk} = B(c_{jk}, R_{jk})$: i.e. $r_{jk} = \max\{R_{jk}, \operatorname{diam}(Q_{jk})\}$. Thanks to Theorem 3.2.2, it is possible to find a vector field $u_{jk} \in H_0^1(B_{jk})$ satisfying $\nabla \cdot u_{jk} = f_{jk} := f_{jk}^+ - f_{jk}^-$, $\|u_{jk}\|_{L^1(B_{jk})} \leq C\theta_{jk}$ and

$$M_{jk} := M_\varepsilon^\alpha(u_{jk}) \leq C\, \{\theta_{jk}^\alpha d_{j+1} + \varepsilon^{\gamma_2} \|f_{jk}\|_{L^2(B_{jk})}^2\}. \tag{3.3.5}$$

Moreover, if $R_{jk} \geq d_{j+1}/2$, the first term in the right-hand side of (3.3.5) can be omitted since one has

$$\theta_{jk}^\alpha d_{j+1} \leq C\varepsilon^{\gamma_2} \|f_{jk}\|_{L^2}^2. \tag{3.3.6}$$

Indeed, in this case, writing $\theta := \theta_{jk}$ and $R := R_{jk}$, one has $\theta^\alpha d_{j+1} \leq 2\theta^\alpha R$ and, using $2 - \alpha = \frac{(1-\gamma)(d+1)}{d-1}$, we get $\theta^\alpha R = [\theta^{\alpha-2} R^{1+d}][\theta^2 R^{-d}] = \varepsilon^{\gamma_2} R^{-d}\theta^2$. Then, (3.3.6) follows from the fact that, by the Cauchy-Schwarz inequality, we have

$$R^{-d}\theta^2 \leq R^{-d}|B_{jk}| \int_{B_{jk}} (f_{jk})^2 \leq C \int_{B_{jk}} (f_{jk})^2.$$

Now, let us define the vector field $u = \sum_{j,k} u_{jk}$, which satisfies

$$\nabla \cdot u = \sum_{j,k} \nabla \cdot u_{jk} = \sum_{j,k} f_{jk} = f := f^+ - f^-.$$

First note that

$$\|u\|_{L^1(Q)} \leq C \sum \|u_{jk}\|_{L^1(B_{jk})} \leq 2C \sum \theta_{jk} = 2C.$$

In order estimate the energy of u, a similar development of $|\sum \nabla u_{jk}|^2$ as in (3.2.13) and the Cauchy-Schwarz inequality give

$$M_\varepsilon^\alpha(u) \leq J \sum_{j=1}^{J} M_\varepsilon^\alpha \left(\sum_{k=1}^{K_j} u_{jk} \right) \leq C\, J \sum_j \left\{ \sum_k M_{jk} + \sum_{(k,l)\in I_j} \sqrt{M_{jk}}\sqrt{M_{jl}} \right\}, \tag{3.3.7}$$

where I_j stands for the set of pairs (k,l) satisfying $k \neq l$, $\theta_{jk} \geq \theta_{jl}$ and $B_{jk} \cap B_{jl} \neq \emptyset$. We have to estimate the two terms in the right-hand side of (3.3.7).

Estimate of the first term in (3.3.7) We recall that $M_{jk} \leq \theta_{jk}^\alpha d_{j+1} + \varepsilon^{\gamma_2}\|f_{jk}\|_{L^2(B_{jk})}^2$. For the second term, note that

$$\textstyle\sum_{j,k} \|f_{jk}\|_{L^2}^2 \leq \|f\|_{L^2}^2. \tag{3.3.8}$$

Indeed, since $f^+ \wedge f^- = 0$, for all j,k, one has $f_{jk}^+ \wedge f_{jk}^- = 0$ as well. In particular, $\|f_{jk}\|_{L^2(B_{jk})}^2 = \|f_{jk}^+\|_{L^2(B_{jk})}^2 + \|f_{jk}^-\|_{L^2(B_{jk})}^2$, $\|f\|_{L^2(Q)}^2 = \|f^+\|_{L^2(Q)}^2 + \|f^-\|_{L^2(Q)}^2$ and (3.3.8) follows from the super-additivity of the power function $x \to |x|^p$ for $p \geq 1$: $|x+y|^p \geq |x|^p + |y|^p$ for $x, y \in \mathbb{R}$ whenever $xy \geq 0$.

For the first term, applying successively the Jensen inequality with power $\alpha \in (0,1)$, the Hölder inequality, (3.3.3) and the fact that $K_j d_{j+1} = C d_{j+1}^{1-d}$ (see (3.3.4)), one gets

$$\begin{aligned}
\sum_{j,k} \theta_{jk}^\alpha d_{j+1} &\leq \sum_j d_{j+1} K_j [\theta_j/K_j]^\alpha = \sum_j [d_{j+1}\theta_j]^\alpha [d_{j+1}K_j]^{1-\alpha} \\
&\leq \left(\sum_j \theta_j\, d_{j+1} \right)^\alpha \left(\sum_j d_{j+1} K_j \right)^{1-\alpha} \\
&\leq C(w+W)^\alpha \left(\sum_j [w(2^{j+1}-1)]^{1-d} \right)^{1-\alpha} \\
&\leq C'(w^\alpha + W^\alpha) w^{(1-d)(1-\alpha)}
\end{aligned}$$

since $\theta_0 d_1 \le d_1 = w$ (we cannot estimate this term by W because $d_0 = 0$) and, because of (3.3.3), $\sum_{j\ge 1}\theta_j\, d_{j+1} \le 3\sum_{j\ge 1}\theta_j\, d_j \le 3W$. Finally, we get

$$\textstyle\sum_{j,k} M_{jk} \le C\left\{w^{1-d(1-\alpha)} + W^\alpha w^{-(d-1)(1-\alpha)} + \varepsilon^{\gamma_2}\|f\|_{L^2}^2\right\}. \tag{3.3.9}$$

Estimate of the second term in (3.3.7) Before following these computations, we need to understand what the condition “$B_{jk}\cap B_{jl} \neq \emptyset$” is meaning. Assume that $(k,l)\in I_j$. From $Q_{jk}\cap Q_{jl}=\emptyset$, we see that either $\operatorname{supp}\rho_{jk}$ or $\operatorname{supp}\rho_l^j$ is not included in Q_{jk} (resp. Q_{jl}). Since, by definition of I_j, we have $\theta_{jk}\ge\theta_{jl}$, this implies that $R_{jk}\ge d_{j+1}/2$. Therefore, as we noticed after formula (3.3.5),

$$M_{jk} \le \varepsilon^{\gamma_2}\|f_{jk}\|_{L^2(B_{jk})}^2$$

and (3.3.6) also implies that

$$\theta_{jl}^\alpha d_{j+1} \le \theta_{jk}^\alpha d_{j+1} \le C\varepsilon^{\gamma_2}\|f_{jk}\|_{L^2(B_{jk})}^2.$$

Now, (3.3.5), the subadditivity of the square root function, the preceding inequality, (3.3.8) and Cauchy-Schwarz inequality give in turn

$$\begin{aligned}
\sum_{(k,l)\in I_j}\sqrt{M_{jk}}\sqrt{M_{jl}} &\le C\sum_{(k,l)\in I_j}\sqrt{\varepsilon^{\gamma_2}\|f_{jk}\|_2^2}\left(\sqrt{\varepsilon^{\gamma_2}\|f_{jl}\|_2^2}+\sqrt{\theta_{jl}^\alpha d_{j+1}}\right)\\
&\le C\varepsilon^{\gamma_2}\sum_{(k,l)\in I_j}\|f_{jk}\|_2^2 + \|f_{jk}\|_2\|f_{jl}\|_2\\
&\le C\varepsilon^{\gamma_2}\left\{K_j\|f_j\|_{L^2(Q)}^2 + \sqrt{\sum_{k,l}\|f_{jk}\|_2^2}\sqrt{\sum_{k,l}\|f_{jk}\|_2^2}\right\}\\
&\le 2C\varepsilon^{\gamma_2}K_j\|f_j\|_{L^2(Q)}^2.
\end{aligned}$$

From $K_j \le d_{j+1}^{-d} \le 2^{-dj}w^{-d}$ and $\|f_j\|_{L^2(Q)}^2 \le \|f\|_{L^2(Q)}^2$, we obtain in the end that

$$\sum_j\sum_{(k,l)\in I_j}\sqrt{M_{jk}}\sqrt{M_{jl}} \le Cw^{-d}\varepsilon^{\gamma_2}\|f\|_{L^2}^2. \tag{3.3.10}$$

End of the proof Let $F=\varepsilon^{\gamma_2}\|f\|_{L^2}^2$. We remind the definition of $W = W_1(f^+,f^-)$. One can assume that $f^-\neq f^+$ so that $F, W>0$. Now, (3.3.7), (3.3.9), (3.3.10) and the fact that $J\le C(1+\ln w)$ yield

$$M_\varepsilon^\alpha(u) \le C(1+|\ln w|)\left\{w^\nu + W^\alpha w^{\nu-\alpha} + w^{-d}F\right\},$$

where $\nu := 1-d(1-\alpha)\in(0,1)$ and so $\alpha-\nu = -(d-1)(1-\alpha)<0$. Let us fix some $\delta\in(0,1)$ small enough so that $0<\nu\pm\delta<1$ and $\nu-\alpha\pm\delta<0$. For some constant c depending on δ, one has $1+|\ln w| \le c(w^\delta + w^{-\delta})$ and so

$$M_\varepsilon^\alpha(u) \le C\left\{w^{\nu\pm\delta} + W^\alpha w^{\nu-\alpha\pm\delta} + w^{-d\pm\delta}F\right\},$$

where the sum is taken over the values of ± 1 ($+1$ or -1) in the right-hand side. Then, we make the choice $w = W + F^\lambda > 0$ for some $\lambda = \lambda(\alpha, d) > 0$ which will be fixed later. Note that all the estimates above are valid only if $w < 1$. However, if $W + F^\lambda \geq 1$ then the right-hand side of (3.3.1) is greater than some positive constant and (3.3.1) easily follows from Theorem 3.2.2 since $H(x) \geq x$. Thus, one can assume that $w \in (0, 1)$.

Since $0 < \nu \pm \delta < 1$, we get $w^{\nu\pm\delta} \leq W^{\nu\pm\delta} + F^{\lambda(\nu\pm\delta)}$ and, because $-d \pm \delta < 0$, $\nu - \alpha \pm \delta < 0$, we have $w^{\nu-\alpha\pm\delta} \leq W^{\nu-\alpha\pm\delta}$ and $w^{-d\pm\delta} \leq F^{\lambda(-d\pm\delta)}$ which gives

$$M_\varepsilon^\alpha(u) \leq C \left\{ W^{\nu\pm\delta} + F^{\lambda(\nu\pm\delta)} + W^{\nu\pm\delta} + F^{1+\lambda(-d\pm\delta)} \right\}.$$

We fix $\lambda > 0$ small enough so that $1 + \lambda(-d \pm \delta) > 0$: in this way, all the exponents in the preceding formula are positive. Finally, (3.3.1) follows from the fact that we have W, $F \leq 1$ as a consequence of W, $F \leq W^{1-d(1-\alpha)} + F$. ☐

Remark 3.3.4. Since $\min\{w^\nu + w^{-d}F \ : \ w \in (0,1)\} = cF^{\frac{1}{d+\nu}}$ and $\frac{1}{d+\nu} < 1$, one cannot obtain an estimate of the form $M_\varepsilon^\alpha(u) \leq C(W + F)$ as expected. However, one could improve a bit (3.3.1) by a better estimate of the number of indices l such that $(k, l) \in I_j$.

Chapter 4

Γ-convergence with divergence constraints

Let $d \geq 1$ and $\Omega \subset \mathbb{R}^d$ a bounded open subset. Let us fix $\mu = \mu^+ - \mu^-$ for two probability measures μ^+ and μ^- compactly supported on Ω. We recall the definition of the set

$$\mathcal{M}_{div}(\Omega) = \{u : \Omega \to \mathbb{R}^d \ : \ u \text{ and } \nabla \cdot u \text{ are finite measures on } \overline{\Omega}\}$$

which is endowed with the topology of weak star convergence on vector measures and their divergence. As weak star topology is never metrizable in infinite dimensional Banach spaces, the space $\mathcal{M}_{div}(\Omega)$ is not metrizable. Indeed, assume that X is some infinite dimensional Banach space such that X' is metrizable. In particular X' admits a countable neighborhood basis $(V_n)_{n\geq 1}$ which one can assume to be of the form

$$V_n = \{\varphi \ : \ |\langle \varphi \, ; x_i \rangle| < \varepsilon_n \text{ for } i = 1, \dots, n\}$$

for some linearly independent family of vectors $(x_i)_{i\geq 1} \subset X$ and $\varepsilon_n > 0$. Then the Hahn-Banach Theorem easily provides a sequence $(\varphi_n)_{n\geq 1}$ satisfying $\varphi_n(x_i) = 0$ for all $i \leq n \in \mathbb{N}^*$ and $\|\varphi_n\|_{X'} = n$. In particular the sequence $(\varphi_n)_n$ weakly converges to 0 as $n \to \infty$ which is a contradiction with the fact that $(\varphi_n)_n$ is norm unbounded.

However, every bounded subsets of the dual space of a separable Banach space are metrizable for the weak star topology. In particular, for the natural norm $\|u\|_{\mathcal{M}_{div}(\Omega)} = |\nabla \cdot u|(\Omega) + |u|(\Omega)$ given by the total variation of u and its divergence, we know that all bounded subsets of $\mathcal{M}_{div}(\Omega)$ are metrizable: for all $M > 0$, there exists a metric d_M for the weak star convergence of u and $\nabla \cdot u$ on the set

$$\mathcal{M}_M(\Omega) = \{u \in \mathcal{M}_{div}(\Omega) \ : \ |u|(\Omega) + |\nabla \cdot u|(\Omega) \leq M\}.$$

In [53] the Γ-convergence of the functional sequence M_ε^α to M^α was proved. Our aim is to prove that this property remains true when adding a divergence constraint. Since, for $u \in H^1(\Omega)$, one has $\nabla \cdot u \in L^2$, one cannot prescribe $\nabla \cdot u = \mu$ if μ is not in L^2. For this reason, we first have to define a regularization of μ. Let $(f_\varepsilon)_{\varepsilon > 0} \subset L^2$ be a sequence of L^2 functions weakly converging to μ as measures and satisfying

$$\int_\Omega f_\varepsilon(x) \, \mathrm{d}x = 0 \quad \text{and} \quad \varepsilon^{\gamma_2} \|f_\varepsilon\|_{L^2}^2 \underset{\varepsilon \to 0}{\longrightarrow} 0 \quad .$$

This choice is going to be useful for the Γ-convergence of M_ε^α under the divergence constraint $\nabla \cdot u = f_\varepsilon$. For example, we can define f_ε as

$$f_\varepsilon := \rho_\varepsilon * \mu,$$

where $\rho_\varepsilon(x) = \varepsilon^{-d\gamma}\rho(\varepsilon^{-\gamma}x)$ for some compactly supported $\rho \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^+)$ such that $\int_\Omega \rho = 1$ and γ is still defined as $\gamma = \frac{\gamma_2}{d+1}$. Now, let us define the functionals $\overline{M}_\varepsilon^\alpha$ (resp $\overline{M}^\alpha$) adding a divergence constraint on $u \in \mathcal{M}_{div}(\Omega)$:

$$\overline{M}^\alpha(u) = \begin{cases} M^\alpha(u) & \text{if } \nabla \cdot u = \mu, \\ +\infty & \text{otherwise.} \end{cases}$$

$$\overline{M}_\varepsilon^\alpha(u) = \begin{cases} M_\varepsilon^\alpha(u) & \text{if } \nabla \cdot u = f_\varepsilon, \\ +\infty & \text{otherwise.} \end{cases}$$

We are going to prove Theorem 1.2.3 (already stated in the introduction):

Theorem. *The functional sequence* $(\overline{M}_\varepsilon^\alpha)_{\varepsilon>0}$ *Γ-converges to* $c_\beta \overline{M}^\alpha$ *as* $\varepsilon \to 0$ *where* c_β *is defined by* (2.1.6) *with* $N = d - 1$.

Note that the $\Gamma-\liminf$ part of the Γ-convergence result stated in Theorem 1.2.3 is a consequence of the Γ-convergence of M_ε^α (without divergence constraint), i.e. Theorem 1.2.2. We have to prove that the $\Gamma - \limsup$ property still is true when adding the divergence constraint. We start by reminding how a recovery sequence for M_ε^α can be constructed and which estimates one can get.

4.1 A fundamental lemma: finding a “nice recovery sequence”

In the proof of Theorem 2.3.1 section 2.3 we showed the existence of a recovery sequence for the energy M_ε^α. Moreover, for vector measures u concentrated on a graph, we obtained some estimates on the extra energy of the recovery sequence $(u_\varepsilon)_{\varepsilon>0}$ we obtained: namely, the difference between $M_\varepsilon^\alpha(u_\varepsilon)$ and $c_\beta M^\alpha(u)$ was of the order of ε^γ. In this section, we are going to estimate the L^2-norm of this recovery sequence. We will see that it diverges, and estimate how much. This will our main tool to prove that we can correct the extra divergence, i.e. find a recovery sequence satisfying the divergence constraint $\nabla \cdot u_\varepsilon = f_\varepsilon$.

We are going to use a construction slightly different from the one which was used in the proof of Theorem 2.3.1 section 2.3. Actually, since we need several estimates on the recovery sequence, it is convenient to build it in a explicit way, that is as a convolution with u at least when it is concentrated on a graph.

Let w be an optimal profile for the problem 2.1.7 in dimension $N := d - 1$. Thanks to Lemma 2.1.1, one can assume that $w \in H^1 \cap \mathrm{Lip}(\mathbb{R}^N, \mathbb{R}^+)$ is compactly supported, smooth on the set $\{w > 0\}$, radially symmetric and that $w(|x'|)$ is non-decreasing with respect to $|x'|$, $x' \in \mathbb{R}^N$. In other words, one can write

$$\text{for a.e.} \quad x' \in \mathbb{R}^N, \quad w(x') = z(|x'|) \tag{4.1.1}$$

for some $z : \mathbb{R}^+ \to \mathbb{R}^+$ compactly supported, Lipschitz on $\mathbb{R}^+$, smooth on the set $\{z > 0\}$, non-increasing and such that $\int_{\mathbb{R}^+} |z'(r)|^2 r^{N-1}\,\mathrm{d}r < +\infty$. We remind the definition of $R_{\theta,\varepsilon} = \varepsilon^\gamma \theta^{\frac{1-\gamma}{N}}$ where $\gamma = \frac{\gamma_2}{d+1}$. Then $v_{\theta,\varepsilon}$, defined by $v_{\theta,\varepsilon}(x) = \theta R_{\theta,\varepsilon}^{-N} w(R_{\theta,\varepsilon}^{-1} x)$, is optimal for the minimization problem (2.1.5), i.e.

$$F_\varepsilon^\beta(v_{\theta,\varepsilon}) = c_\beta \theta^\alpha = \inf\left\{ F_\varepsilon^\beta(v) \ : \ v \in H^1(\mathbb{R}^N, \mathbb{R}^+) \quad \text{and} \quad \int_{\mathbb{R}^N} v = \theta \right\}.$$

As before, we use the system of coordinates $x = (x_1, x') \in \mathbb{R}^d$. We start by proving the existence of a kernel $\rho_{\theta,\varepsilon}$ whose projection on the hyperplane $(x_1 = 0)$ is equal to the optimal profile $v_{\theta,\varepsilon}$:

Lemma 4.1.1. *There exists a bounded and compactly supported radial kernel* $\rho_{\theta,\varepsilon} \in L_c^\infty(\mathbb{R}^d, \mathbb{R}^+)$ *such that* $v_{\theta,\varepsilon}$ *is the projection of* $\rho_{\theta,\varepsilon}$ *on the hyperplane* $(x_1 = 0)$*:*

$$\Pi_\# \rho_{\theta,\varepsilon}(x)\,\mathrm{d}x = v_{\theta,\varepsilon}(x')\,\mathrm{d}x',$$

where Π *stands for the orthogonal projection on the variable* $x' \in \mathbb{R}^N$ *and* $\mathrm{d}x$ *(resp.* $\mathrm{d}x'$*) is the Lebesgue measure on* $\mathbb{R}^d$ *(resp.* $\mathbb{R}^N$, $N = d-1$*). Moreover one can choose* $\rho_{\theta,\varepsilon}$ *of the form* $\rho_{\theta,\varepsilon}(x) = R_{\theta,\varepsilon}^{-d} \rho(R_{\theta,\varepsilon}^{-1} x)$ *for some* $\rho \in L_c^\infty(\mathbb{R}^d, \mathbb{R}^+)$*.*

Remark 4.1.2. Since $\rho_{\theta,\varepsilon}$ and $v_{\theta,\varepsilon}$ are radially symmetric, the choice of the hyperplane we consider, here $(x_1 = 0)$, has no importance.

Proof. First renormalize the problem writing $\rho_{\theta,\varepsilon}(x) = R_{\theta,\varepsilon}^{-d} \rho(R_{\theta,\varepsilon}^{-1} x)$ so that it is enough to find ρ satisfying

$$\Pi_\# \rho(x) dx = w(x')\,\mathrm{d}x' \quad . \tag{4.1.2}$$

Let $z : \mathbb{R}^+ \to \mathbb{R}^+$ be the function defined in (4.1.1). Since w is smooth around 0 and has a maximum at the origin, one has $\nabla w(0) = 0$ and so $z'(0) = 0$. We are going to see that a radial solution of (4.1.2) is given by the formula

$$\rho(x) = f(|x|) \quad \text{where} \quad f(r) = \int_r^\infty \frac{-z'(s)}{\pi\sqrt{s^2 - r^2}}\,\mathrm{d}s \quad . \tag{4.1.3}$$

For the sake of simplicity, we will denote by $\Pi\rho \in L^1(\mathbb{R}^N)$ the function such that $\Pi_\# \rho\,\mathrm{d}x = \Pi\rho(x')\,\mathrm{d}x'$ for any $\rho \in L^1(\mathbb{R}^d)$. Note that

$$\Pi\rho(x') = \int_{\mathbb{R}} \rho(x_1, x')\,\mathrm{d}x' \quad \text{a.e.}$$

We have to solve the equation $\Pi\rho = w$. We look for a function ρ of the form $\rho(x) = f(|x|)$ for some function $f : \mathbb{R}^+ \to \mathbb{R}^+$ which shares the properties of z: f is continuous, nonnegative, non-increasing and compactly supported. We will denote by X the space of functions satisfying these properties. Note that the continuity of f is a necessary condition for ρ to be in H^1. Now, in order to find ρ such that $\Pi\rho = w$, it is enough to find $f \in X$ such that

$$\forall r \geq 0,\ z(r) = \int_{\mathbb{R}} f(\sqrt{t^2 + r^2})\,\mathrm{d}t \quad .$$

Then the function defined by $\rho(x) = f(|x|)$ satisfies $\Pi\rho = w$. Let denote by T the operator corresponding to the preceding operation. The change of variable $s = \sqrt{t^2 + r^2}$ yields the following expression for T:

$$\forall r \geq 0,\ Tf(r) := 2\int_0^{\infty} f(\sqrt{t^2 + r^2})\,\mathrm{d}t = \int_0^{\infty} f(s)K(s,r)\,\mathrm{d}s \quad ,$$

where $K(s,r) = \frac{2s\mathbf{1}_{s>r}}{\sqrt{s^2-r^2}}$. Then T is a continuous linear operator from $L^1(\mathbb{R}^+)$ to $L^1_{loc}(\mathbb{R}^+)$ (for the family of seminorms $|\cdot|_{L^1([0,R])}$, $R > 0$). Indeed, for all $R > 0$, one has

$$\begin{aligned}
\int_0^R |Tf(r)|\,\mathrm{d}r &\leq \int_0^R \int_r^{2r} |f|(s)K(s,r)\,\mathrm{d}s\,\mathrm{d}r + \int_0^R \int_{2r}^{\infty} |f|(s)K(s,r)\,\mathrm{d}s\,\mathrm{d}r \\
&\leq \int_0^{2R} \left(\int_{s/2}^{s} K(s,r)\,\mathrm{d}r \right) |f|(s)\,\mathrm{d}s + \sup_{0 \leq 2r \leq s} K(s,r)\ \|f\|_{L^1} \\
&\leq \left\{ \sup_{0 \leq s \leq 2R} \int_{s/2}^{s} K(s,r)\,\mathrm{d}r + \sup_{0 \leq 2r \leq s} K(s,r) \right\} \|f\|_{L^1} \\
&\leq CR\|f\|_{L^1}
\end{aligned}$$

for some $C > 0$, where we used the inequality $K(s,r) \leq 4$ for $s \geq 2r$ and the equality $\int_{s/2}^{s} K(s,r)\,\mathrm{d}r = \int_{1/2}^{1} \frac{2s\,\mathrm{d}r}{\sqrt{1-r^2}} = c_0 s$.

Now we have to compute the inverse of the operator T. Let us define a function $k_\lambda : \mathbb{R}^+ \to \mathbb{R}$ by $k_\lambda(r) = \lambda^{-1}k(\lambda^{-1}r)$ where

$$k(r) = \begin{cases} \frac{1}{\pi\sqrt{1-r^2}} & \text{if } r < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $Tk = \mathbf{1}_{[0,1]}$ and so $Tk_\lambda = \mathbf{1}_{[0,\lambda]}$. Indeed, if $|r| > 1$, $Tk(r) = 0$ while, if $|r| \leq 1$,

$$Tk(r) = 2\int_0^{\sqrt{1-r^2}} \frac{dt}{\pi\sqrt{1 - t^2 - r^2}} = 2\int_0^1 \frac{dt}{\pi\sqrt{1-t^2}} = 1.$$

Now it easy to compute f such that $Tf = z$ if z is $\mathcal{C}^1_c$: for all $r \geq 0$,

$$\begin{aligned}
z(r) &= -\int_r^{\infty} z'(s)\,\mathrm{d}s = -\int_0^{\infty} \mathbf{1}_{[0,s]}(r)z'(s)\,\mathrm{d}s \\
&= -\int_0^{\infty} T[z'(s)k_s(\cdot)](r)\,\mathrm{d}s = T\left[-\int_0^{\infty} z'(s)k_s\,\mathrm{d}s \right](r)
\end{aligned}$$

which implies the claim. Note that the exchange between T and the integration is well justified. Indeed, one can think of the last two lines in the preceding equations as a Riemann integral in the space $L^1(\mathbb{R}^+)$ or $L^1_{loc}(\mathbb{R}^+)$. Since $s \mapsto z'(s)k_s \in L^1(\mathbb{R}^+)$ is continuous and since T is a continuous linear operator, the integrals are well defined and commute with T.

Since we are not sure that $z \in \mathcal{C}^1$ in dimension $N \geq 2$, these computations are not always legitimate. However, for sure, z is Lipschitz continuous. Then one can smooth z by convolution (one can first extend z to an even function on $\mathbb{R}$) and obtain a sequence $(z_n)_{n\geq 1} \subset X \cap \mathcal{C}^\infty$ uniformly converging to z and such that $(z'_n)_n$ is bounded in L^∞ and converges to z' in L^1. Moreover, since z is compactly supported, one can assume that the supports of z_n are all included in a fixed compact set . Now, for each $n \geq 1$, one can define f_n by the formula (4.1.3). Let us denote by $\rho_n \in H^1(\mathbb{R}^d)$ (resp. $w_n \in H^1(\mathbb{R}^N)$) the functions corresponding to f_n (resp. z_n): $\rho_n(x) = f_n(|x|)$ and $w_n(x) = z_n(|x|)$. Then, by construction one has

$$\Pi_{\#}\rho_n(x)\,\mathrm{d}x = w_n(x')\,\mathrm{d}x'$$

and it is not difficult to pass to the limit when $n \to \infty$. Indeed, since the sequence $(w_n)_n$ converges uniformly, it converges strongly in L^1 and ρ_n is a Cauchy sequence in L^1. Let $\rho \in L^1$ denote the strong limit in L^1. Up to extraction, one can assume that $(\rho_n)_n$ converges almost everywhere. In particular, ρ shares all the properties of w: $\rho(x) = f(|x|)$ for some $f \in X$.

It remains to prove that f is bounded. Note that, since z'_n converges to z' in L^1, we know that f and z are related each other by the second equation of (4.1.3). One can deduce that f is bounded. Indeed, since $z'(0) = 0$ and since z' is bounded on $\mathbb{R}^+$ and smooth around 0, we know that $|z'(s)| \leq Cs$ for all $s \geq 0$ and some constant $C > 0$. Hence, there exists $C > 0$ such that for all $r \in [0, R)$,

$$f(r) = \int_1^R \frac{-z'(rs)\,\mathrm{d}s}{\pi\sqrt{s^2-1}} \leq C\int_1^{R/r} \frac{rs\,\mathrm{d}s}{\sqrt{s^2-1}} \leq C\left\{R\int_1^2 \frac{s\,\mathrm{d}s}{\sqrt{s^2-1}} + r\int_2^{R/r} \frac{s\,\mathrm{d}s}{\sqrt{s^2-1}}\right\}$$

which is bounded since $s \to \frac{s}{\sqrt{s^2-1}}$ is integrable on $[1, 2]$ and bounded on $[2, +\infty)$. □

As a consequence, in the case where $u = \theta\mathcal{H}^1 \llcorner S$, a recovery sequence, i.e. a sequence (u_ε) such that $u_\varepsilon \to u$ in $\mathcal{M}_{div}(\Omega)$ and $M^\alpha_\varepsilon(u_\varepsilon) \to M^\alpha(u)$ as $\varepsilon \to 0$, is obtained as

$$u_\varepsilon = \rho_{\theta,\varepsilon} * u \quad .$$

In the case of a finite energy configuration, i.e. $u \in \mathcal{M}_{div}(\Omega)$ such that $M^\alpha(u) < \infty$, thanks to classical properties in the theory of Γ-convergence, it is enough to find a recovery sequence for u belonging to a class of measures which are dense in energy. Thanks to Proposition 1.1.6, we know that the class of vector measures concentrated on finite graphs is dense in energy so that one can restrict to this case. This was used in chapter 2 to prove the Γ-convergence of M^α_ε toward M^α. In the setting of functionals with divergence constraint, we need the following lemma:

Lemma 4.1.3. *Let $u \in \mathcal{M}_{div}(\Omega)$ be such that $M^\alpha(u) < \infty$. For all $\lambda > \frac{d\gamma}{2}$, there exists a sequence $(u_\varepsilon) \subset H^1_0(\Omega)$ converging to u in $\mathcal{M}_{div}(\Omega)$ such that*

$$M^\alpha_\varepsilon(u_\varepsilon) \xrightarrow[\varepsilon\to 0]{} c_\beta M^\alpha(u) \quad \textit{and} \quad \varepsilon^\lambda\|\nabla\cdot u_\varepsilon\|_{L^2} \quad \textit{is bounded.}$$

Before proving this statement, we are going to investigate the case where u is concentrated on a finite graph, that is when u is a transport path. For the definition of a "transport path", we refer to section 1.1. In particular, we keep the same notation, that is

$$u_G := \sum_{\mathbf{e}=(a_{\mathbf{e}},b_{\mathbf{e}})\in E(g)} \theta(\mathbf{e})\tau_{\mathbf{e}}\,\mathrm{d}\mathcal{H}^1 \llcorner e$$

for any weighted oriented graph $G = (E(G), \theta)$. When u is a transport path, we have the following lemma:

Lemma 4.1.4. *Let $u = u_G \in \mathcal{M}_{div}(\Omega)$ for some weighted directed graph G. Then, there exists a sequence $(u_\varepsilon)_{\varepsilon>0}$ converging to u in $\mathcal{M}_{div}(\Omega)$ and a constant C depending on u such that, for ε small enough, $u_\varepsilon \in H^1_0(\Omega)$ and*

1. $\int_\Omega |u_\varepsilon| \le |u|(\Omega) + C\,\varepsilon^\gamma$,
2. $\int_\Omega |\nabla\cdot u_\varepsilon| \le |\nabla\cdot u|(\Omega)$,
3. $\varepsilon^{\frac{d\gamma}{2}}\,\|\nabla\cdot u_\varepsilon\|_{L^2} \le C$,
4. $|M^\alpha_\varepsilon(u_\varepsilon) - c_\beta M^\alpha(u)| \le C\varepsilon^\gamma$.

Note that, in this case, we are able to prove that $\varepsilon^{\frac{d\gamma}{2}}\,\|\nabla\cdot u_\varepsilon\|_{L^2}$ (rather than $\varepsilon^\lambda\,\|\nabla\cdot u_\varepsilon\|_{L^2}$ for $\lambda > \frac{d\gamma}{2}$) is bounded.

Proof. By definition, such a vector measure u can be written as a finite sum of measures $u_i = \theta_i\,\tau_i\,\mathcal{H}^1 \llcorner S_i$ concentrated on a segment $S_i \subset \Omega$ directed by τ_i with multiplicity θ_i for $i = 1, \dots, I$. We first define a regularized vector fied v_ε by $v_\varepsilon := \sum_i v_i$, where $v_i = \rho_{\theta_i,\varepsilon} * u_i$. Then, for ε small enough, v_ε is compactly supported on Ω and satisfies

$$\begin{cases} \int_\Omega |v_\varepsilon| \le |u|(\Omega), \\ |M^\alpha_\varepsilon(v_\varepsilon) - c_\beta M^\alpha(u)| \le C\varepsilon^\gamma. \end{cases}$$

The first statement is a consequence of the fact that $\int \rho_{\theta_i,\varepsilon} = 1$ and the inequality $\|f * \mu\|_{L^1} \le \|f\|_{L^1}\,|\mu|(\Omega)$ for $f \in \mathcal{C}_c(\Omega)$ and for a finite measure μ on Ω. For the second statement, by definition of the kernel $\rho_{\theta,\varepsilon}$ we know that, out of the nodes set $\mathcal{N} = \bigcup_i \operatorname{supp}(\nabla\cdot v_i)$,

$$M^\alpha_\varepsilon(v_\varepsilon, \mathcal{N}^c) = c_\beta M^\alpha(v, \mathcal{N}^c).$$

As a result, we just have to estimate these energies on $\mathcal{N}$ which is a finite union of balls: the supports of $\rho_{\theta_i,\varepsilon}$ recentered at each end-point of the segment S_i. Since the radius of these balls is of the order of ε^γ, this immediately gives the fact that $M^\alpha(u, \mathcal{N}) \le C\varepsilon^\gamma$ for some constant $C > 0$ depending on u. For the sake of simplicity, in the rest of this proof, $C > 0$ will denote some constant depending on u which is large enough so that all the inequalities below are true. We are going to prove that

$$M^\alpha(u, \mathcal{N}) + M^\alpha_\varepsilon(v_\varepsilon, \mathcal{N}) \le C\varepsilon^\gamma.$$

It remains to estimate $M^\alpha_\varepsilon(v_\varepsilon, \mathcal{N})$. Since $M^\alpha_\varepsilon(v_\varepsilon, \mathcal{N}) \le I \sum_i M^\alpha_\varepsilon(v_i, \mathcal{N})$, it is enough to estimate $M^\alpha_\varepsilon(v_i, \mathcal{N})$. But $\|v_i\|_{L^\infty(\mathcal{N})} = \|\rho_{\theta_i,\varepsilon} * u_i\|_{L^\infty(\mathcal{N})} \le C\varepsilon^{-d\gamma}\|\rho\|_{L^\infty}|u_i|(\mathcal{N}_i)$, where $\mathcal{N}_i := \mathcal{N} + \operatorname{supp}\rho_{\theta_i,\varepsilon} := \{x + y \;:\; x \in \mathcal{N},\, y \in \operatorname{supp}\rho_{\theta_i,\varepsilon}\}$. Note that $\operatorname{supp}\rho_{\theta_i,\varepsilon}$ is a ball centered at the origin with radius smaller than $C\varepsilon^\gamma$ so that $\mathcal{N}_i$ is a finite union of balls with radii smaller than $C\varepsilon^\gamma$ as well. In particular, using the fact that

$u_i = \theta_i \, \tau_i \, \mathcal{H}^1 \llcorner S_i$, we get $|u_i|(\mathcal{N}_i) \le C\varepsilon^{\gamma}$ and so $\|v_i\|_{L^\infty(\mathcal{N})} \le C\varepsilon^{(1-d)\gamma}$. Similarly, one has $\|\nabla v_i\|_{L^\infty(\mathcal{N})} = \|\nabla \rho_{\theta_i,\varepsilon} * u_i\|_{L^\infty(\mathcal{N})} \le C\varepsilon^{-(d+1)\gamma}\|\nabla\rho\|_{L^\infty}|u_i|(\mathcal{N}_i) \le C\varepsilon^{-d\gamma}$. Now, the definition (1.2.2) gives

$$\begin{aligned} M_\varepsilon^\alpha(v_\varepsilon, \mathcal{N}) &= \int_{\mathcal{N}} \varepsilon^{\gamma_2}|\nabla v_\varepsilon|^2 + \varepsilon^{-\gamma_1}|v_\varepsilon|^\beta \,\mathrm{d}x \\ &\le |\mathcal{N}|\{\varepsilon^{\gamma_2}\|\nabla v_\varepsilon\|_{L^\infty}^2 + \varepsilon^{-\gamma_1}\|v_\varepsilon\|_{L^\infty}^\beta\} \\ &\le C\varepsilon^{d\gamma}\{\varepsilon^{\gamma_2 - 2d\gamma} + \varepsilon^{-\gamma_1-(d-1)\beta\gamma}\} \\ &\le 2C\varepsilon^{\gamma}. \end{aligned}$$

for some constant $C > 0$ independant of ε, where the following elementary computations on exponents γ, γ_1, γ_2 and β have been used:

$$\begin{cases} d\gamma + \gamma_2 - 2d\gamma = \gamma & , \\ d\gamma - \gamma_1 - (d-1)\beta\gamma = \gamma & . \end{cases}$$

We still need to impose the second and third properties. Actually, v_ε will not satisfy them (for instance, the divergence of v_ε does not vanish at the nodes even if $\nabla \cdot u = 0$), but we can replace it by $u_\varepsilon := v_\varepsilon - w_\varepsilon$ where $w_\varepsilon \in H_0^1(\mathcal{N})$ is constructed as follows:

The node set is a finite union $\mathcal{N} = \bigcup_{j=1}^n B_j$, where each node B_j is a ball centered at the end-point a_i of some segment $S_i = [a_i, b_i]$. Let us assume that ε is small enough so that these balls are non-overlapping. Then, on each node B_j, $g_j := \nabla \cdot v_\varepsilon$ is a finite superposition of kernels like $\rho_{\theta,\varepsilon}$ recentered at c_j, the center of B_j. In particular $\|g_j\|_{L^2} \le C\varepsilon^{-\frac{d\gamma}{2}}$ and $\int_{B_j} g_j = \int_{B_j} \nabla \cdot v_\varepsilon = (\nabla \cdot u)(B_j) =: \theta_j$.

If $\theta_j = 0$, then Theorem 3.2.4 allows to find $w_j \in H_0^1(B_j)$ satisfying $\nabla \cdot w_j = g_j$ and $\|w_j\|_{H^1(B_j)} \le C\,\varepsilon^{-\frac{d\gamma}{2}}$.

If $\theta_j \ne 0$, say $\theta_j > 0$, we rewrite g_j as $g_j = g^+ - g^- = \lambda g^+ + (1-\lambda)g^+ - g^-$ where g^+ (resp. g^-) stands for the positive part (resp. negative part) of g and $\lambda \in (0,1]$ is chosen such that $(1-\lambda)\int_B g^+ = \int_B g^-$, i.e. $\theta_j = \lambda \int_{B_j} g_+$. Applying Theorem 3.2.4, we get $w_j \in H_0^1(B_j)$ satisfying $\nabla \cdot w_j = (1-\lambda)g^+ - g^-$ and $\|w_j\|_{H^1(B_j)} \le C\,\varepsilon^{-\frac{d\gamma}{2}}$. Note that $\int_{B_j} |g_j - \nabla \cdot w_j| = \lambda \int_{B_j} g^+ = \theta_j$.

Now, let us define $w_\varepsilon = \sum_j w_j$ and $u_\varepsilon := v_\varepsilon - w_\varepsilon$. Since $\int_{B_j} |\nabla \cdot u_\varepsilon| = \int_{B_j} |g_j - \nabla \cdot w_j| = \theta_j$ for all j, we have

$$\int_\Omega |\nabla \cdot u_\varepsilon| = \int_{\mathcal{N}} |\nabla \cdot u_\varepsilon| \le \sum_j \theta_j = |\nabla \cdot u|(\Omega).$$

Moreover, to estimate $\|\nabla \cdot u_\varepsilon\|_{L^2}$, note that $|\nabla \cdot w_\varepsilon|_{L^2} \le |w_\varepsilon|_{H^1} \le C\varepsilon^{-\frac{d\gamma}{2}}$ and, because $\nabla \cdot v_\varepsilon$ is only composed of a finite sum of translated kernels of the form $\rho_{\theta_i,\varepsilon}$, $\|\nabla \cdot v_\varepsilon\|_{L^2} \le C\varepsilon^{-\frac{d\gamma}{2}}$ as well. In particular $\varepsilon^{\frac{d\gamma}{2}}\|\nabla \cdot u_\varepsilon\|_{L^2}$ is bounded. Then, from a Sobolev inequality, we deduce that

$$\|w_\varepsilon\|_{L^2} = \sum_j \|w_j\|_{L^2(B_j)} \le C \sum_j \varepsilon^{\gamma}\|\nabla w_j\|_{L^2} \le C'\varepsilon^{(1-\frac{d}{2})\gamma}.$$

since the radius of B_j is of the order of ε^γ. Consequently, by the Cauchy-Schwarz inequality, we get

$$\int_\Omega |u_\varepsilon| \leq \int_\Omega |v_\varepsilon| + \int_{\mathcal{N}} |w_\varepsilon| \leq |u|(\Omega) + |\mathcal{N}|^{1/2}\|w_\varepsilon\|_{L^2} \leq |u|(\Omega) + C\varepsilon^\gamma.$$

Similarly, by a Hölder inequality, we have

$$\int_{\mathcal{N}} |w_\varepsilon|^\beta \leq \varepsilon^{d\gamma(1-\frac{\beta}{2})}\|w_\varepsilon\|_{L^2}^\beta \leq C\varepsilon^{d\gamma(1-\frac{\beta}{2})+\beta\gamma(1-\frac{d}{2})} = C\varepsilon^{\gamma(d+\beta-\beta d)}.$$

Once again, by definition $\gamma = \frac{\gamma_2}{d+1}$. Moreover, quite easy computations give $\beta\gamma = 4\gamma - 2$ and we deduce

$$M_\varepsilon^\alpha(w_\varepsilon) = \varepsilon^{\gamma_2}\int_{\mathcal{N}} |\nabla w_\varepsilon|^2 + \varepsilon^{-\gamma_1}\int_{\mathcal{N}} |w_\varepsilon|^\beta \leq C\{\varepsilon^{\gamma_2 - d\gamma} + \varepsilon^{-\gamma_1+\gamma(\beta+d-\beta d)}\} = 2C\varepsilon^\gamma.$$

Since $M^\alpha(u,\mathcal{N}) \leq C\varepsilon^\gamma$, we get $M_\varepsilon^\alpha(u_\varepsilon,\mathcal{N}) \leq 2[M_\varepsilon^\alpha(v_\varepsilon,\mathcal{N}) + M_\varepsilon^\alpha(w_\varepsilon,\mathcal{N})] \leq C\varepsilon^\gamma$ which finally gives

$$|M_\varepsilon^\alpha(u_\varepsilon) - c_\beta M^\alpha(u)| = |M_\varepsilon^\alpha(u_\varepsilon,\mathcal{N}) - c_\beta M^\alpha(u,\mathcal{N})| \leq C\varepsilon^\gamma. \quad \square$$

Proof of Lemma 4.1.3. First fix a vector field $u \in \mathcal{M}_{div}(\Omega)$ and construct a sequence $(u_n)_{n\geq 1}$ converging to u such that $u_n = u_{G_n}$ is a vector measure associated to some weighted directed graph $G_n \subset \Omega$ and $M^\alpha(u_n)$ converges to $M^\alpha(u)$. Since (u_n) weakly converges in $\mathcal{M}_{div}(\Omega)$, the total variations of both measures u_n and $\nabla \cdot u_n$ are bounded by some constant $M > 0$. In the following, we use a metric d on the space $\mathcal{M}_{M+1}(\Omega)$. Extracting a subsequence if necessary, one can suppose that the two following estimates hold

$$d(u_n, u) \leq 2^{-n-1} \quad \text{and} \quad |M^\alpha(u_n) - M^\alpha(u)| \leq 2^{-n-1}.$$

For each $n \geq 1$, let $u_{\varepsilon,n}$ be a sequence converging to u_n as $\varepsilon \to 0$ and satisfying all properties in Lemma 4.1.4 for some constant $C = C_n$. Then, one can construct by induction a decreasing sequence $(\varepsilon_n)_{n\geq 1} \to 0$ such that for all $n \geq 1$ and $\varepsilon \leq \varepsilon_n$, $u_{\varepsilon,n} \in H_0^1(\Omega)$ and

1. $u_{\varepsilon,n} \in \mathcal{M}_{M+1}(\Omega)$,
2. $d(u_{\varepsilon,n}, u_n) \leq 2^{-n-1}$,
3. $|M_\varepsilon^\alpha(u_{\varepsilon,n}) - M^\alpha(u_n)| \leq 2^{-n-1}$,
4. $\varepsilon^{\lambda - \frac{d\gamma}{2}} C_n \leq 1$ so that $\varepsilon^\lambda \|\nabla \cdot u_{\varepsilon,n}\|_{L^2} \leq 1$.

Indeed, assume that $\varepsilon_n > 0$ satisfies all the asked properties. Then, one can find $\varepsilon_{n+1} \in (0, \varepsilon_n)$ small enough so that

- $*$ $C_{n+1}\,\varepsilon_{n+1}^\gamma < 2^{-n-2}$ thus implying the first and third properties (see properties 1., 2. and 4. in Lemma 4.1.4),
- $*$ $C_{n+1}\,\varepsilon_{n+1}^{\lambda-\frac{d\gamma}{2}} < 1$ which is possible since $\lambda > \gamma$
- $*$ and $d(u_{\varepsilon,n+1}, u_{n+1}) \leq 2^{-n-2}$ for all $\varepsilon \in (0, \varepsilon_{n+1})$ which is possible since $u_{\varepsilon,n+1}$ converges to u_{n+1} in $(\mathcal{M}_{M+1}(\Omega), d)$ as $\varepsilon \to 0$.

Now it is quite straightforward that the sequence $(u_\varepsilon)_{\varepsilon>0}$ defined by

$$u_\varepsilon = \begin{cases} u_{\varepsilon,1} & \text{if } \varepsilon > \varepsilon_1, \\ u_{\varepsilon,n} & \text{if } \varepsilon_{n+1} < \varepsilon \leq \varepsilon_n \text{ for some } n \geq 1, \end{cases}$$

satisfies all properties of Lemma 4.1.3. □

4.2 Upper bound with divergence constraints: proof of Theorem 1.2.3

We have already shown in chapter 2 that $M_\varepsilon^\alpha \xrightarrow{\Gamma} c_\beta M^\alpha$. We just have to prove that the $\Gamma - \limsup$ property still holds when we add the divergence constraint. In other words, it remains to prove that for all $u \in \mathcal{M}_{div}(\Omega)$ such that $\nabla \cdot u = \mu$, there exists a sequence $(v_\varepsilon)_{\varepsilon>0} \subset \mathcal{M}_{div}(\Omega)$ weakly converging to u as measures, satisfying $\nabla \cdot v_\varepsilon = f_\varepsilon$ (so that (v_ε) also converges in $\mathcal{M}_{div}(\Omega)$) and $M_\varepsilon^\alpha(v_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} M^\alpha(u)$.

First of all, take a sequence $(u_\varepsilon)_{\varepsilon>0} \subset H_0^1(\Omega)$ converging to u given by Lemma 4.1.3 for some λ such that $\frac{d\gamma}{2} < \lambda < \frac{\gamma_2}{2}$ (this is possible since $\frac{d\gamma}{2} = \frac{d\gamma_2}{2(d+1)} < \frac{\gamma_2}{2}$). Namely, one has $M_\varepsilon^\alpha(u_\varepsilon) \to c_\beta M^\alpha(u)$ with $\varepsilon^\lambda \|\nabla \cdot u_\varepsilon\|_{L^2}$ bounded. Then define $g_\varepsilon = g_\varepsilon^+ - g_\varepsilon^- := f_\varepsilon - \nabla \cdot u_\varepsilon$ the residual divergence. In particular, $\int_\Omega g_\varepsilon = 0$. Indeed, one has $\int_\Omega f_\varepsilon = \int_\Omega \nabla \cdot u_\varepsilon = 0$ since $u_\varepsilon \in H_0^1(\Omega)$. Moreover, our hypothesis on the sequence $(f_\varepsilon)_{\varepsilon>0}$, that is $\varepsilon^{\gamma_2}\|f_\varepsilon\|_{L^2}^2 \underset{\varepsilon\to 0}{\longrightarrow} 0$, and the fact that $\varepsilon^\lambda\|\nabla\cdot u_\varepsilon\|_{L^2}$ is bounded (with $2\lambda < \gamma_2$) yield $\varepsilon^{\gamma_2}\|\nabla \cdot u_\varepsilon\|_{L^2}^2 \underset{\varepsilon\to 0}{\longrightarrow} 0$ and then

$$\varepsilon^{\gamma_2}\|g_\varepsilon\|_{L^2}^2 \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Moreover, since f_ε and $\nabla\cdot u_\varepsilon$ weakly converge to μ as ε goes to 0, we know that g_ε weakly converges to 0. In particular, $W_1(g_\varepsilon^+, g_\varepsilon^-) \underset{\varepsilon\to 0}{\longrightarrow} 0$ (see the comments below (1.1.5)). In order to satisfy the divergence constraint, we may "correct" u_ε with a vector field w_ε. Applying Theorem 3.3.1 (together with Remark 3.3.2), we get w_ε satisfying $\nabla \cdot w_\varepsilon = g_\varepsilon$ and

$$M_\varepsilon^\alpha(w_\varepsilon) \leq H\left(W_1(g_\varepsilon^+, g_\varepsilon^-)^{1-d(1-\alpha)} + \varepsilon^{\gamma_2}\|g_\varepsilon\|_{L^2}^2\right) \underset{\varepsilon\to 0}{\longrightarrow} 0\,, \quad \|w_\varepsilon\|_{\mathcal{M}_{div}(\Omega)} \leq C \quad , \tag{4.2.1}$$

where $H(x) = C(x + x^\delta)$ for some $C > 0$ and $\delta \in (0,1)$. In particular, (w_ε) is relatively compact in $\mathcal{M}_{div}(\Omega)$. Now, (4.2.1) and the $\Gamma - \liminf$ property imply that w_ε converges to 0 in $\mathcal{M}_{div}(\Omega)$. Last of all, by construction, $v_\varepsilon = u_\varepsilon + w_\varepsilon$ satisfies $\nabla \cdot v_\varepsilon = f_\varepsilon$, $v_\varepsilon \to u$ in $\mathcal{M}_{div}(\Omega)$ and $M_\varepsilon^\alpha(v_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} M^\alpha(u)$. Indeed, this last limit is a consequence of

Lemma 4.2.1. *Let Ω be some bounded open set in $\mathbb{R}^d$, $d \geq 1$. Let (u_ε), $(v_\varepsilon) \subset H^1(\Omega)$ be two sequences such that $M_\varepsilon^\alpha(u_\varepsilon - v_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 0$ and assume that $M_\varepsilon^\alpha(v_\varepsilon)$ is bounded. Then,*

$$|M_\varepsilon^\alpha(u_\varepsilon) - M_\varepsilon^\alpha(v_\varepsilon)| \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Proof. Let $\nu > 0$ be some constant. For all real matrices A and B of size $d \times d$, by the Young inequality, we have

$$|A+B|^2 = |A|^2 + |B|^2 + 2A : B \leq (1+\nu)|A|^2 + (1+1/\nu)|B|^2.$$

Writing $u_\varepsilon = v_\varepsilon + u_\varepsilon - v_\varepsilon$, we use the preceding inequality for $A = \nabla v_\varepsilon$, $B = \nabla(u_\varepsilon - v_\varepsilon)$ and the subadditivity of $x \to |x|^\beta$ to get

$$M_\varepsilon^\alpha(u_\varepsilon) = \varepsilon^{-\gamma_1} \int_\Omega |u_\varepsilon|^\beta + \varepsilon^{\gamma_2} \int_\Omega |\nabla u_\varepsilon|^2 \leq (1+\nu) M_\varepsilon^\alpha(v_\varepsilon) + (1+1/\nu) M_\varepsilon^\alpha(u_\varepsilon - v_\varepsilon).$$

Since $M_\varepsilon^\alpha(v_\varepsilon) < C$ for some constant $C < +\infty$, we deduce that

$$M_\varepsilon^\alpha(u_\varepsilon) - M_\varepsilon^\alpha(v_\varepsilon) \leq C\nu + (1+1/\nu) M_\varepsilon^\alpha(u_\varepsilon - v_\varepsilon).$$

For any value of ε such that $u_\varepsilon \neq v_\varepsilon$, let us take $\nu = \sqrt{M_\varepsilon^\alpha(u_\varepsilon - v_\varepsilon)} > 0$. Hence, taking the lim sup when $\varepsilon \to 0$, one gets

$$\limsup_{\varepsilon \to 0} \{M_\varepsilon^\alpha(u_\varepsilon) - M_\varepsilon^\alpha(v_\varepsilon)\} \leq C' \limsup_{\varepsilon \to 0} \sqrt{M_\varepsilon^\alpha(u_\varepsilon - v_\varepsilon)} = 0.$$

Since $M_\varepsilon^\alpha(v_\varepsilon) \leq 2[M_\varepsilon^\alpha(u_\varepsilon) + M_\varepsilon^\alpha(v_\varepsilon - u_\varepsilon)]$ and $M_\varepsilon^\alpha(v_\varepsilon)$ is bounded, we deduce that $M_\varepsilon^\alpha(u_\varepsilon)$ is bounded as well. Then we can apply all the preceding computations exchanging u_ε and v_ε to get $\limsup_{\varepsilon \to 0} \{M_\varepsilon^\alpha(v_\varepsilon) - M_\varepsilon^\alpha(u_\varepsilon)\} \leq 0$ which concludes the proof. ☐

Conclusion and perspectives

The main problematic of this chapter was the Γ-convergence of functionals M_ε^α in every dimension and, possibly, with a divergence constraint. Our method, based on the analysis of scalar Cahn-Hilliard models for droplets equilibrium allows to prove the Γ-convergence in the case $1 - \frac{1}{d} < \alpha < 1$ without divergence constraint. However, the case $0 < \alpha \leq 1 - \frac{1}{d}$ remains open. Actually, if $0 < \alpha < 1 - \frac{1}{d}$, then the corresponding exponent β is negative. Hence, the potential t^β, singular at $t = 0$, has to be replaced by some function $W(t)$ which behaves as t^β for large values of t (see (2.1.3)). That work was done in dimension 2 in [53]. In higher dimension, the question remains open. The main issue is to prove that the limiting energy is local. In other words, it should be proved that using a cut-off function does not impact a lot the energy M_ε^α. Our hope is that the energy is local even for $\alpha \leq 1 - \frac{1}{d}$. Note that G. Bouchitté, C. Dubs and P. Seppecher encountered the same difficulty to prove the locality of their model for Cahn-Hilliard fluids (see [29, 15]) which corresponds, somehow, to the energy M_ε^α on each slice. Namely, we do not know whether the limiting energy of Cahn-Hilliard type models, defined in (2.1.3), is local in dimension $N \geq 3$ and for $\beta < 0$. Since functionals M_ε^α in dimension $d \geq 2$ are related to Cahn-Hillard models in dimension $N = d - 1$, it seems that the locality for functional M_ε^α could also be proved in dimension 3. Another property established in this chapter concerns the validity of the Γ-convergence theorem with a divergence constraint when $1 - \frac{1}{d} < \alpha < 1$. Once again, we think that the result remains true up to performing some modifications on the potential W, as before, when $0 < \alpha \leq 1 - \frac{1}{d}$. A perspective which could simplify the proof of the lower bound and might extend it to the case $0 < \alpha \leq 1 - \frac{1}{d}$ would be to use a dual method analogous to the calibration method for minimal surfaces. We point out the fact that this work was done in [1] for the Mumford-Shah functional which has some similarities with our approximations M_ε^α.

The main property which is lacking in this chapter is the compactness for functionals M_ε^α: that is the relative compactness for finite energy sequences. This is a fundamental property which is required for a Γ-convergence result to be useful, for instance for numerical applications. Here the natural topology is given by the weak convergence of vector measures and there divergence. Nevertheless, it is not difficult to see that the compactness is false in our context. Indeed one can build sequences $(u_\varepsilon)_\varepsilon$ which have bounded energy, $M_\varepsilon^\alpha(u_\varepsilon) \leq C < \infty$, but which are unbounded in L^1. This compactness property actually fails even for the limiting energy M^α in the space of measures. An alternative is to consider a weaker topology, for instance, the $W^{-1,p}$ topology, for some $p \geq 1$. Some discussion with G. Bouchitté, A. Julia and F. Santambrogio suggested that there could exist $p \geq 1$ such that the compactness holds in $W^{-1,p}$. Another alternative is to show the compactness only for minimizing sequences $(u_\varepsilon)_\varepsilon$. More precisely, we consider a sequence $(f_\varepsilon)_\varepsilon$, with $\int f_\varepsilon = 0$, weakly converging to f as measures. Then, considering a minimizing sequence $(u_\varepsilon)_\varepsilon$ of M_ε^α under the constraint $\nabla \cdot u_\varepsilon = f_\varepsilon$, we wonder whether $(u_\varepsilon)_\varepsilon$ is relatively compact in $\mathcal{M}_{div}$.

Part II

Aviles-Giga models: 1D symmetry, semicontinuity and entropies

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MODÈLES D'AVILES-GIGA : SYMMÉTRIE 1D, SEMICONTINUITÉ ET ENTROPIES

Résumé

Dans cette partie, nous étudions les couches limites dans des modèles de type Aviles-Giga principalement en dimension 2. Nous considérerons une énergie faisant intervenir un terme de Dirichlet et un potentiel non-convexe s'annulant sur une ou plusieurs courbes régulières. Ce type d'énergies, définies pour des champs de vecteurs à divergence nulle, présentent de nombreux exemples notamment en théorie du micromagnétisme. Dans un régime de faible intéraction, c'est à dire quand le terme associé au potentiel l'emporte sur le terme de Dirichlet, on observe que la symmétrie des minimiseurs dans les couches limites dépend fortement du potentiel. Nous allons essayer de comprendre quelles conditions il faut imposer sur le potentiel afin que les minimiseurs globaux soient à symmétrie $1D$, c'est à dire ne dépendant que d'une seule variable.

Abstract

In this part, we analyze the boundary layers in some Aviles-Giga type models, principally in dimension 2. We consider some energy composed of two terms, a Dirichlet term and a non-convex potential vanishing along one or more smooth curves. This kind of energies, defined over divergence-free vector fields, has many examples in micromagnetics. In the low-interaction regime, that is when the potential term is stronger than the Dirichlet term, the symmetry of minimizers turns out to be strongly related to the potential. We will try to understand which conditions may be imposed on the potential for the global minimizers to be one-dimensional, that is only depending on a single variable.

Structure of this part In the first chapter, we analyze the line-energies which appears as limit of Aviles-Giga type energies. These free discontinuity energies are defined over divergence-free BV-functions as the integral of some function, named *cost function*, of the jump size. We point out a necessary condition on the cost function for the minimizers under boundary constraints to be one-dimensional symmetric and for the corresponding line-energy to be lower semicontinuous. In the second chapter, we analysis the one-dimensional symmetry of the optimal profile of some Aviles-Giga type energies under strong assumptions on the potential. The main tool will be the entropy method which was introduced by P. Aviles and Y. Giga. In the last chapter, we apply the entropy method and the tools of the previous chapter to deduce a Γ-convergence result in every dimension, namely the $\Gamma-\liminf$ bound, on these energies.

Chapter 5

Introduction

5.1 General framework

The model In this part, we will consider functionals of the form

$$E(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla u|^2 + W(u)\,\mathrm{d}x & \text{if } \nabla \cdot u = 0, \\ +\infty & \text{otherwise,} \end{cases} \tag{5.1.1}$$

defined for vector fields $u : \Omega \subset \mathbb{R}^d \to \mathbb{R}^d$ where $W : \mathbb{R}^d \to \mathbb{R}^+$ is a nonnegative potential and Ω some open set. The first term is called *Dirichlet energy* or *exchange/interaction energy* in the micromagnetics jargon, while the second term is called *potential energy.* Except some results in every dimension and some examples in dimension $d = 3$, we will mostly consider the two dimensional case. Many functionals of this type can be found in the theory of elasticity, liquid crystals or micromagnetics. A fundamental example is due to P. Aviles and Y. Giga (see [7]) who studied some functional of the form (5.1.1) in two dimensions and for the Ginzburg-Landau potential $W(z) = (1 - |z|^2)^2$. In dimension $d = 2$, it is interesting to see that (5.1.1) is a kind of second order Allen-Cahn or Modica-Mortola functional for gradient fields. Indeed, any divergence-free vector field $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ is the rotated gradient of some scalar function $\varphi : \Omega \to \mathbb{R}$, the so-called stream function. Then, the energy E_ε reads

$$E(u) = F(\varphi) := \frac{1}{2}\int_\Omega |\nabla^2 \varphi|^2 + G(\nabla\varphi)\,\mathrm{d}x$$

whenever $u = (\nabla\varphi)^\perp$ where $G(z) := W(z^\perp)$ for all $z \in \mathbb{R}^2$. Note that the definition of F does not involve any constraint anymore since the divergence constraint is encoded in the expression of u as a rotated gradient.

Low interaction regime We are particularly interested in the regime where the Dirichlet energy $\int_\Omega |\nabla u|^2$ is less penalized than the potential energy $\int_\Omega W(u)$. Namely, given some small parameter $\varepsilon > 0$, we consider the energy

$$E_\varepsilon(u, \Omega) = E_\varepsilon(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega \varepsilon|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\,\mathrm{d}x & \text{if } \nabla \cdot u = 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{5.1.2}$$

If there is ambiguity on the set Ω where u is defined, we will prefer the notation $E_\varepsilon(u,\Omega)$ rather than $E_\varepsilon(u)$: if u is defined one some open set $\omega \supset \Omega$, we define $E_\varepsilon(u,\Omega)$ by (5.1.2).

If the potential W vanishes on a nontrivial set S, called the *well set*, finite energy configurations will concentrate on S as ε goes to 0. As it happens for the classical scalar Modica-Mortola functional $F_\varepsilon(\psi) = \frac{1}{2}\int \varepsilon|\nabla\psi|^2 + \frac{1}{\varepsilon}W(\psi)$, the Dirichlet energy penalizes the variations of u between different values in S. In other words, the transition between two wells $u^\pm$ costs some positive energy depending on $u^\pm$ and W. In the easiest situation, W vanishes on a discrete set S. This is the case of some thin-film micromagnetics models where a finite number of values for the magnetization (called easy axis) are favored. We will also consider examples of potentials W which vanish on a finite union of smooth lines in $\mathbb{R}^d$. This makes the study of the asymptotic behavior of the functional E_ε much more arduous since very complex admissible structures can appear in the limit when ε goes to 0. For the Ginzburg-Landau potential, this was pointed out by P. Aviles, Y. Giga in [7] and W. Jin, R. V. Kohn in [39]. They introduced a class of energy functionals defined over unit divergence-free vector fields, the so-called line energies which appear as singular limit of the "Aviles-Giga functional" (corresponding to $d = 2$ and $W(z) = (1-|z|^2)^2$).

Γ-convergence Many questions arise from this kind of models concerning the asymptotic behavior of the energy E_ε and the minimizers (under boundary constraints) when $\varepsilon \to 0$. This kind of questions in the calculus of variations have been formalized by E. De Giorgi. Let us outline the principal definitions and properties of the Γ-convergence theory, as introduced by E. De Giorgi (see [18] or [24] for further study).

Definition 5.1.1. Let (X,d) be a metric space and $(F_\varepsilon)_{\varepsilon>0} : X \to \mathbb{R}\cup\{+\infty\}$ a sequence of functionals defined on X. We say that the sequence $(F_\varepsilon)\varepsilon > 0$ Γ-converges to $F : X \to \mathbb{R}\cup\{+\infty\}$ and we note $F = \Gamma - \lim_{\varepsilon\to 0} F_\varepsilon$ if the following two properties hold.

Lower bound: for all sequence $(x_\varepsilon)_{\varepsilon>0} \subset X$ converging to $x \in X$ as $\varepsilon \to 0$, one has

$$F(x) \leq \liminf_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon). \tag{5.1.3}$$

Upper bounded: for all $x \in X$, there exists a sequence $(x_\varepsilon)_{\varepsilon>0}$ converging to x as $\varepsilon \to 0$ such that

$$F(x) = \lim_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon), \tag{5.1.4}$$

which means that the lower bound (5.1.3) is sharp. Independently, we define the $\Gamma-\liminf$ of the sequence $(F_\varepsilon)_{\varepsilon>0}$ as

$$\Gamma - \liminf_{\varepsilon\to 0} F_\varepsilon(x) := \inf\{\liminf_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) \ : \ x_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} x\} \quad \text{for all } x \in X$$

and the $\Gamma - \limsup$ of the sequence $(F_\varepsilon)_{\varepsilon>0}$ as

$$\Gamma - \limsup_{\varepsilon\to 0} F_\varepsilon(x) := \inf\{\limsup_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) \ : \ x_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} x\} \quad \text{for all } x \in X.$$

Clearly, the sequence $(F_\varepsilon)_{\varepsilon>0}$ Γ-converges to F if and only if one has

$$F = \Gamma - \liminf_{\varepsilon\to 0} F_\varepsilon = \Gamma - \limsup_{\varepsilon\to 0} F_\varepsilon \quad .$$

The sequence $(F_\varepsilon)_{\varepsilon>0}$ is said *equi-coercive* on X is for all $R>0$ there exists a compact subset $K\subset X$ such that

$$\forall\varepsilon>0,\ \{x\in X\ :\ F_\varepsilon(x)\le R\}\subset K.$$

Among all the properties of the Γ-convergence, we point out the following:

Proposition 5.1.2. *Let $(F_\varepsilon)_{\varepsilon>0}$ be a sequence Γ-converging to $F:X\to\mathbb{R}\cup\{+\infty\}$.*

- **Semicontinuity of Γ-limits** *F is lower semicontinuous.*
- **Existence of minimizers** *Assume furthermore that the both following conditions are satisfied,*
  - **Compactness:** *every bounded energy sequence $(x_\varepsilon)_{\varepsilon>0}\subset X$, i.e.*

    $$\sup\{F_\varepsilon(x_\varepsilon)\ :\ \varepsilon>0\}<+\infty,$$

    is compact in (X,d). This is the case, for instance, when the sequence $(F_\varepsilon)_{\varepsilon>0}$ is equi-coercive.
  - **Finiteness:** $\inf\limits_X F>-\infty$.

  Then F attains its minimum and $\min\limits_X F=\lim\limits_{\varepsilon\to0}\inf\limits_X F_\varepsilon$.
- **Stability of minimizers** *Let $(x_\varepsilon)_{\varepsilon>0}$ be a sequence of minimizers for F_ε admitting a subsequence converging to x, then x minimizes F.*
- **Stability of Γ-convergence** *Let $G:X\to\mathbb{R}\cup\{+\infty\}$ be a continuous functional, then $(F_\varepsilon+G)_{\varepsilon>0}$ Γ-converges to $F+G$.*

Concerning the Γ-limit of E_ε (defined in (5.1.2)), if it exists, only partial results are known for specific potentials W. When W vanishes on a non trivial connected set S, the question of the Γ-convergence of the sequence $(E_\varepsilon)_{\varepsilon>0}$ as well as the compactness property are challenging problems. Given a potential W vanishing on $S\subset\mathbb{R}^d$, the sequence $(E_\varepsilon)_{\varepsilon>0}$ is expected to Γ-converge in $L^1(\Omega,\mathbb{R}^d)$ to some free discontinuity energy functional of the form

$$E_f(u)=\begin{cases}\displaystyle\int_{J(u)} f\left(u^+(x),u^-(x),\nu_u(x)\right)\mathrm{d}\mathcal{H}^{d-1}(x) & \text{if } \nabla\cdot u=0 \text{ and } u\in S \text{ a.e. },\\ +\infty & \text{otherwise,}\end{cases}$$

for all $u\in\mathrm{BV}(\Omega,\mathbb{R}^d)$ with jump set $J(u)$ oriented by ν_u and traces $u^\pm$ on each side of $J(u)$ w.r.t to ν_u, where $f:\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}\to\overline{\mathbb{R}}:=\mathbb{R}^+\cup\{+\infty\}$ is some nonnegative l.s.c. function, called *cost function.* Note that the non-convex term $\int W(u)$ in the energy E_ε (which turns into the non-convex constraint $W(u)=0$ in the limit) justifies the choice of the L^1 strong convergence for the Γ-convergence of E_ε. Before giving examples of models related to (5.1.2), we are going to sketch what we mean by free discontinuity energy by giving a precise definition of E_f.

5.2 Free discontinuity problems

We quickly recall how free discontinuity problems are formulated in the set of bounded variation functions, BV. We refer to [30] or [5] for a detailed study of BV functions and applications to free discontinuity problems. Since we are also interested in applications in dimension 3, we give a definition in every dimension.

Free discontinuity energy Given $u \in \mathrm{BV}(\Omega, \mathbb{R}^d)$, $\mathrm{D}u$ is a finite measure that decomposes in

$$\mathrm{D}u = \mathrm{D}^a u + \mathrm{D}^j u + \mathrm{D}^c u \quad , \tag{5.2.1}$$

where $\mathrm{D}^a u = \nabla^a u \, \mathcal{L}^d$ is the absolute continuous part of $\mathrm{D}u$ (∇u is the approximate gradient of u), $\mathrm{D}^c u$ is the Cantor part of $\mathrm{D}u$ and $\mathrm{D}^j u$ is the jump part. We refer to [30] and [5] (Definition 3.92, page 184) for more details. In particular, we have

$$\mathrm{D}^j u = (u^+ - u^-) \otimes \nu_u \, \mathrm{d}\mathcal{H}^{d-1} \llcorner J(u),$$

where $J(u)$ is the jump set of u, ν_u its orientation and $u^\pm$ the traces of u on each size of $J(u)$ oriented by ν_u. $J(u)$ is a $\mathcal{H}^{d-1}$-rectifiable subset of $\mathbb{R}^d$ and ν_u is a measurable function from $J(u)$ to $\mathbb{S}^{d-1}$ such that $\nu_u(x)$ is orthogonal to $J(u)$ for $\mathcal{H}^{d-1}$-a.e. $x \in J(u)$. Moreover, $u^\pm(x)$ are determined by the formula

$$\lim_{r \to 0} \frac{1}{r^d} \int_{B_\nu^\pm(x,r)} |u(y) - u^\pm(x)| \, \mathrm{d}y = 0 \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } x \in J(u), \tag{5.2.2}$$

where $B_\nu^\pm(x,r) := \{y \in B(x,r) \ : \ \pm y \cdot \nu_u(x) \geq 0\}$. We call free discontinuity problems some minimization problems where the energy concentrates on the jump set $J(u)$. The free energy of some configuration u depends on $u^\pm$ and the normal vector ν_u. Let $f : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1} \to \overline{\mathbb{R}} = \mathbb{R}^+ \cup \{+\infty\}$ be a nonnegative borelian function satisfying the following invariance property:

$$\forall u^\pm \in \mathbb{R}^d, \forall \nu \in \mathbb{S}^{d-1}, \ f(u^+, u^-, \nu) = f(u^-, u^+, -\nu) = f(u^+, u^-, -\nu) \ . \tag{5.2.3}$$

The function f is called *cost function.* For every $u \in \mathrm{BV}(\Omega, \mathbb{R}^d)$, the free energy of u is defined as

$$E_f(u, \Omega) = E_f(u) = \begin{cases} \displaystyle\int_{J(u)} f\left(u^+(x), u^-(x), \nu_u(x)\right) \mathrm{d}\mathcal{H}^{d-1}(x) & \text{if } \nabla \cdot u = 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{5.2.4}$$

As before we prefer the notation $E_f(u, \Omega)$ rather that E_f if there is ambiguity on the set Ω and we afford to write $E_f(u, \Omega)$ (defined by (5.2.4)) when u is defined on some set $\omega \supset \Omega$. Note that E_f is well defined, i.e. $E_f(u)$ does not depend on the choice for the orientation ν and the corresponding traces $u^\pm$. Indeed, assume that $(\overline{u}^+, \overline{u}^-, \overline{\nu}) : \Omega \to \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}$ is another representation of the jump part of u, i.e. $\overline{\nu}$ is orthogonal to $J(u)$ a.e. and $\overline{u}^\pm$ satisfy (5.2.2). Then, for a.e. $x \in J(u)$,

$$(\overline{u}^+(x), \overline{u}^-(x), \overline{\nu}(x)) = (u^+(x), u^-(x), \nu(x)) \text{ or } (u^-(x), u^+(x), -\nu(x))$$

so that $f(u^+(x), u^-(x), \nu(x)) = f(\overline{u}^-(x), \overline{u}^+(x), \overline{\nu}(x))$ because of the invariance assumption (5.2.3). Since we are interested in free energies which are Γ-limit of some of the functional sequences $(E_\varepsilon)_{\varepsilon>0}$, the question of the lower semicontinuity of E_f is fundamental. Indeed any Γ-limit is necessarily l.s.c. In chapter 6, we will see that very little is known about the l.s.c. of this kind of energies even in dimension 2, i.e. for line energies. The only necessary condition that easily follows is the l.s.c. of the cost function f.

If u is divergence-free, the triplet (u^-, u^+, ν) cannot be any element of the set $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}$. Indeed, (u^-, u^+, ν) must fulfill some conditions, given by the following lemma:

Lemma 5.2.1. *Assume that* $u \in \mathrm{BV}(\Omega, \mathbb{R}^d)$ *is divergence-free. Then one has*

$$u^+(x) \cdot \nu_u(x) = u^-(x) \cdot \nu_u(x) \quad \mathcal{H}^{d-1}\text{-}a.e. \ on \ J(u).$$

Proof. Let $x \in J(u)$ such that (5.2.2) holds. Let us take $r > 0$ such that $B(x, r) \subset \Omega$ and define the blow-up of u at x by

$$u_r(x') = u(x + rx')$$

for all $x' \in B(0, 1)$. Then u_r is divergence-free on $B(0, 1)$ and u_r converges in L^1 to u_0 as $r \to 0$, where

$$u_0(x') = \begin{cases} u^+ & \text{if } x' \cdot \nu_u(x) \geq 0, \\ u^- & \text{if } x' \cdot \nu_u(x) < 0. \end{cases}$$

In particular u_0 is divergence-free which implies the claim. Indeed, $u_0 \in \mathrm{BV}$ and $\nabla u_0 = (u^+ - u^-) \otimes \nu \, \mathrm{d}\mathcal{H}^{d-1}$ so that $\nabla \cdot u_0 = (u^+ - u^-) \cdot \nu = 0$. □

Line energies In dimension $d = 2$, thanks to Lemma 5.2.1, every divergence-free vector field $u \in \mathrm{BV}(\Omega, \mathbb{R}^2)$ satisfies

$$\nu_u(x) = \pm \frac{[u^+(x) - u^-(x)]^\perp}{|u^+ - u^-|} \quad .$$

Moreover, the second invariance property of (5.2.3) implies that $f(u^+, u^-, \nu)$ only depends on u^+, u^- and the direction of ν: $\pm\nu$. Thus, in dimension 2, (5.2.4) simply reads

$$\mathcal{I}_f(u) = \int_{J(u)} f(u_+, u_-) \, \mathrm{d}\mathcal{H}^1(x)$$

for some cost function $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^+ \cup \{+\infty\}$.

Exemple 5.2.2. A fundamental example of singular energies of this type was introduced by P. Aviles and Y. Giga in [7] and has already been mentioned above. The authors obtained these so called line energies as singular limit of Ginzburg-Landau type energies. Namely, they considered the energy

$$AG_\varepsilon(u) = \frac{1}{2} \int_\Omega \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon}(1 - |u|^2)^2$$

defined for divergence-free vector fields $u : \Omega \to \mathbb{R}^2$, where $\Omega \subset \mathbb{R}^2$ is some bounded domain. In the limit when $\varepsilon \to 0$, they obtained a line energy of the form $\mathcal{I}_f$ for the cubic cost:

$$f(u^+, u^-) = \begin{cases} \frac{|u^+ - u^-|^3}{6} & \text{if } u^\pm \in \mathbb{S}^1, \\ +\infty & \text{otherwise.} \end{cases}$$

Note that, in this case, f is finite only on the set $\{u^\pm \in \mathbb{S}^1\}$ and one could consider f as a function on $(\mathbb{S}^1)^2$. In general, thinking of E_f as the limit of some energies E_ε defined by (7.1.1) for some potential W, one could consider f as a function on the set $\{(u^+, u^-, \nu) \ : \ W(u^\pm) = 0 \text{ and } u^+ \cdot \nu = u^- \cdot \nu\}$. Indeed, admissible configurations for the asymptotic energy, i.e. limits of finite energy sequences for E_ε, may satisfy $W(u^\pm) = 0$ a.e. Moreover, in the Aviles-Giga case, f only depends on $|u^- - u^+|$ which is natural since W is invariant by rotation.

Although we have chosen to restrict to BV functions, the space BV is not always pertinent for all potentials W. For instance, in [4], the authors gave an example of microstructure $u \notin \mathrm{BV}(\Omega)$ which is admissible for the limiting energy associated with AG_ε, i.e. $u = \lim_{\varepsilon \to 0} u_\varepsilon$ in L^1 with $\sup_\varepsilon \mathrm{AG}_\varepsilon(u_\varepsilon) < \infty$. In [27], the authors gave an alternative candidate for set of admissible limiting configurations. Namely, they introduced a functional set $\mathcal{AG}$ containing $\mathrm{BV}(\Omega, \mathbb{R}^2)$ such that $\mathcal{AG}$ contains every limits of bounded energy sequences.

5.3 Cost function associated to the potential

We want to generalize the situation of Aviles and Giga: if $d = 2$ and $W(z) = (1 - |z|^2)^2$, the limiting energy of the functional sequence AG_ε is expected to be of the form $\mathcal{I}_f$ with the cubic cost f. In general, it is very easy to estimate E_ε from below using a one-dimensional analysis.

Assume that $W : \mathbb{R}^d \to \mathbb{R}^+$ is nonnegative and continuous. The *singular set*, or *well set*, is denoted by S and defined by $S := \{z \in \mathbb{R}^d \ : \ W(z) = 0\}$. Let us fix two wells $u^\pm \in S$. Our aim is to compute the energy for the transition from u^- to u^+ assuming that the one-dimensional transition layer is optimal. In other words, assuming that $(E_\varepsilon)_{\varepsilon > 0}$ Γ-converges to some free energy E_f and given $u \in \mathrm{BV}(\Omega, \mathbb{R}^d)$, we make the ansatz that a recovery sequence can be chosen in such a way that $u_\varepsilon(x) \approx u_\varepsilon(\mathrm{dist}(x, J(u)))$ at least for $\mathrm{dist}(x, J(u)) \ll 1$.

More precisely, for all ONB (orthonormal basis) $(\nu_1, \nu_2, \ldots, \nu_d)$ of $\mathbb{R}^d$, let us define

$$H^1_{per}(u^+, u^-, (\nu_i)_i) = \Big\{ u \in H^1_{loc}(\mathbb{R}^d, \mathbb{R}^d) \ : \ \nabla \cdot u = 0,\ u(x) = u^\pm \text{ for } \pm x_1 \geq 1/2 \\ \text{and } u(x + \nu_i) \text{ for } x \in \mathbb{R}^d \text{ and } i = 2, \ldots, d \Big\}.$$

Note that, for all $u \in H^1_{per}(u^+, u^-, (\nu_i)_i)$, one has $\nabla \cdot u = 0$ and so $u^+ \cdot \nu_1 = u^- \cdot \nu_1$. Conversely, $H^1_{per}(u^+, u^-, (\nu_i)_i) \neq \emptyset$ whenever $u^+ \cdot \nu_1 = u^- \cdot \nu_1$: one can choose u of the form $u(x) = \varphi(x \cdot \nu)$ with $\varphi_1 \equiv cte$ and $\varphi(t) = u^\pm$ for $\pm t \geq 1/2$. In brief, one has

$$H^1_{per}(u^+, u^-, (\nu_i)_i) \neq \emptyset \Leftrightarrow u^+ \cdot \nu_1 = u^- \cdot \nu_1 \quad . \tag{5.3.1}$$

Then we define the periodic cost as the minimal energy of E_ε over the preceding set:

$$\begin{aligned} c_W^{per}(u^+,u^-,\nu) := \inf\Big\{ \liminf_{\varepsilon\to 0} E_\varepsilon(u,\Omega(\nu_1,\cdots,\nu_d)) \ :\ u\in H^1_{per}(u^+,u^-,(\nu_i)_i),\\ (\nu_i)_i \text{ ONB s.t. } \nu_1=\nu\Big\}, \end{aligned} \tag{5.3.2}$$

where $\Omega(\nu_1,\cdots,\nu_d) := \{x\in\mathbb{R}^d\ :\ |x\cdot\nu_i|\le 1/2 \text{ for all } i=1,\dots,d\}$. In view of (5.3.1), one has

$$c_W^{per}(u^+,u^-,\nu) < +\infty \Leftrightarrow u^+\cdot\nu = u^-\cdot\nu \quad .$$

Analogously, we define $c_W^{1D}(u^-,u^+,\nu)$ as the infimum of the energy over the one-dimensional transition layers, i.e. s.t. $u(x)=\varphi(x\cdot\nu)$ for some $\varphi:\mathbb{R}\to\mathbb{R}^d$. Note that, for vector fields of the preceding form, the divergence constraint, $\nabla\cdot u=0$, turns into $\varphi'(t)\cdot\nu=0$ for a.e. $t\in\mathbb{R}$ and so $\varphi\cdot\nu = cte$. Thus the one-dimensional cost reads

$$\begin{aligned} c_W^{1D}(u^+,u^-,\nu) := \inf\Big\{ \liminf_{\varepsilon\to 0} \frac{1}{2}\int_{-1/2}^{1/2} \varepsilon|\varphi'(t)|^2 + \frac{1}{\varepsilon}W(\varphi(t))\,\mathrm{d}t\ :\\ \varphi\in H^1_{loc}(\mathbb{R},\mathbb{R}^d) \text{ s.t. } \forall t\in\mathbb{R},\ \varphi(t)\cdot\nu = a \text{ and } \varphi(\pm 1/2)=u^\pm\Big\}. \end{aligned} \tag{5.3.3}$$

Of course, for all $u^\pm\in\mathbb{R}^d$, $\nu\in\mathbb{S}^{d-1}$, one has

$$c_W^{per}(u^+,u^-,\nu) \le c_W^{1D}(u^+,u^-,\nu) \quad . \tag{5.3.4}$$

It is not difficult to see that c_W^{1D} actually corresponds (when it is finite) to the geodesic distance between u^- and u^+ in $H(\nu,a) := \{z\in\mathbb{R}^d\ :\ z\cdot\nu = a\}$ endowed with the (singular) riemannian metric $g_W := Wg$ where g is the standard Euclidean metric in $H(\nu,a)\sim\mathbb{R}^{d-1}$.

Proposition 5.3.1. *Assume that $u^\pm\in H(\nu,a)$. Then one has*

$$c_W^{1D}(u^+,u^-,\nu) = \inf\left\{ \int_I |\gamma'(t)|\sqrt{W(a,\gamma(t))}\,\mathrm{d}t\ :\ I\subset\mathbb{R} \text{ connected, } \gamma\in H^1(I,H(\nu,a))\right\}.$$

In particular, if $d=2$, then $H(\nu,a)$ is a straight line and so the change of variable $\gamma=\gamma(t)$ yields

$$c_W^{1D}(u^+,u^-,\nu) = c_W^{1D}(u^+,u^-) = \int_{u_2^-}^{u_2^+} \sqrt{W(a,y)}\,\mathrm{d}y \quad .$$

Proof. We refer to section 7.1.2 (see Proposition 7.1.2) for a proof. The main idea is to use a Young inequality in order to estimate c_W^{1D} from below: let us fix φ as in the definition of c_W^{1D} and $\gamma:\mathbb{R}\to H(\nu,a)$ such that $\varphi(t)=(a,\gamma(t))$. In particular $\gamma(t)=u_2^\pm$ for $\pm t\ge 1/2$ and

$$\begin{aligned} \frac{1}{2}\int_{\mathbb{R}} \varepsilon|\varphi'(t)|^2 + \frac{1}{\varepsilon}W(\varphi(t))\,\mathrm{d}t &= \frac{1}{2}\int_{-1}^{1} \varepsilon|\gamma'(t)|^2 + \frac{1}{\varepsilon}W(a,\gamma(t))\,\mathrm{d}t \quad ,\\ &\ge \int_{-1}^{1} |\gamma'(t)|\sqrt{W(a,\gamma(t))}\,\mathrm{d}t \quad . \end{aligned}$$

Moreover, the inequality is sharp whenever $\varepsilon|\varphi'(t)|^2 = \frac{1}{\varepsilon}W(\varphi(t))$ a.e. □

Many questions arise concerning the periodic and one-dimensional costs. Let us point out a few of them:

Question 1: For which potentials W (5.3.4) becomes an equality, i.e.

$$c_W^{per}(u^+,u^-,\nu) = c_W^{1D}(u^+,u^-,\nu) \text{ for all } u^\pm,\ \nu \quad ? \tag{5.3.5}$$

Question 2: For which potentials W, we have that the infimum value in (5.3.3) or (5.3.2) is achieved ? Is the global minimizer, in case it exists, unique ?

Question 3: For which potentials W, we have that E_f is a lower bound of E_ε in the sense of (5.1.3) for $f = c_W^{per}$? $f = c_W^{1D}$? Of course, if (5.1.3) holds for $f = c_W^{per}$ then it holds for $f = c_W^{1D}$ since $c_W^{per} \leq c_W^{1D}$.

Question 4: Is the upper bound property (5.1.4) true when $F_\varepsilon = E_\varepsilon$ and $F = E_f$ for $f = c_W^{per}$? $f = c_W^{1D}$? As before, it is clear that, if $\Gamma - \limsup E_\varepsilon(u) \leq E_f(u)$ is satisfied for $f = c_W^{per}$, then it is satisfied for $f = c_W^{1D}$.

In chapter 6, we give a necessary condition (CN) on cost functions f of the from $g(|u^+ - u^-|)$ in dimension 2, for the first property, $c_W^{per} = c_W^{1D}$, to be satisfied (see Corollary 6.1.10). To this aim, we will give $2D$-structures which cost less energy than the one-dimensional transition layer when (CN) fails. In chapter 7, we give a partial answer to the first and second questions. The last chapter is concerned with the lower bound. Under some strong conditions on the potential, we will prove that the lower bound is achieved by the one-dimensional transition layer (i.e. for $f = c_W^{1D}$) at least for limiting functions in $\mathrm{BV}(\Omega,\mathbb{R}^d)$. Concerning the upper bound, i.e. Question 4, quite general results are known for limiting BV structures. In [23], S. Conti and C. De Lellis have shown the upper bound for the Aviles-Giga functional. This result has been generalized by A. Poliakovsky in [55]. In particular, one has (see Theorem 1.1. in [55]):

Theorem 5.3.2. *Let $\Omega \subset \mathbb{R}^d$ be an open set and $W : \mathbb{R}^d \to \mathbb{R}^+$ be a $\mathcal{C}^1$ function. Assume furthermore that $u \in \mathrm{BV} \cap L^\infty(\mathbb{R}^d,\mathbb{R}^d)$ satisfies $\nabla \cdot u = 0$ and $|Du|(\partial\Omega) = 0$. Then, there exists a sequence $(u_\varepsilon)_{\varepsilon>0} \subset \mathcal{C}^\infty(\mathbb{R}^d,\mathbb{R}^d)$ such that $u_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} u$ in L^p for every $p \in [1,\infty)$ and*

$$E_\varepsilon(u_\varepsilon,\Omega) \underset{\varepsilon\to 0}{\longrightarrow} E_f(u,\Omega),$$

where $f = c_W^{per}$.

Unfortunately, the upper bound inequality remains open for limiting configurations $u \notin \mathrm{BV}$. In general, it is conceivable that the energy also concentrates on the Cantor part of the derivative of u (the support of $D^c u$ in (5.2.1)) for non BV structures $u \in L^1(\Omega)$.

5.4 Related models

5.4.1 Aviles-Giga functional

The Aviles-Giga functional is one of the most common example of functional of the form (5.1.2). In [7] and [39], the authors have produced the following result:

Theorem 5.4.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set. We consider the following functional, defined for $u \in L^1(\Omega, \mathbb{R}^2)$ by*

$$AG_\varepsilon(u) = \begin{cases} \frac{1}{2}\int_\Omega \varepsilon|\nabla u|^2 + \frac{1}{\varepsilon}(1-|u|^2)^2 & \textit{if } u \in H^1(\Omega,\mathbb{R}^2) \textit{ and } \nabla \cdot u = 0, \\ +\infty & \textit{otherwise,} \end{cases} \tag{5.4.1}$$

where $\varepsilon > 0$ is some parameter. For every sequence $(u_\varepsilon)_{\varepsilon>0} \subset L^1(\Omega, \mathbb{R}^2)$ strongly converging to $u \in \mathrm{BV}(\Omega, \mathbb{R}^2)$ in L^1, one has

$$E_f(u) \leq \liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon), \tag{5.4.2}$$

where $f : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{S}^1 \mapsto \mathbb{R}^+ \cup \{+\infty\}$ is defined by

$$f(u^+, u^-, \nu) := \begin{cases} \frac{|u^+-u^-|^3}{6} & \textit{if } u^\pm \in \mathbb{S}^1, \\ +\infty & \textit{otherwise.} \end{cases}$$

Moreover, E_f is l.s.c. for the L^1 convergence: for every sequence $(u_n)_{n\geq 1} \subset \mathrm{BV}(\Omega, \mathbb{R}^2)$ strongly converging to $u \in \mathrm{BV}$ in L^1, one has

$$E_f(u) \leq \liminf_{n\to\infty} E_f(u_n) \quad .$$

If $u \notin \mathrm{BV}$, one may replace E_f, only defined on $\mathrm{BV}(\Omega, \mathbb{R}^2)$, by its relaxation $\overline{E}_f$ for the L^1 convergence in (5.4.2). In [27], the authors propose a natural candidate for the admissible set $\{u \in L^1(\Omega, \mathbb{R}^2) \ : \ \overline{E}_f(u) < +\infty\}$ (see chapter 6 or [27] for further explanations).

The strong compactness of finite energy sequences has been proved by Ambrosio, De Lellis and Mantegazza in [4] and by De Simone, Kohn, Müller and Otto in [28] using a compensated compactness method based on a new notion of regular entropy on $\mathbb{R}^2$.

Theorem 5.4.2. *Let $(u_\varepsilon)_{\varepsilon>0} \subset L^1(\Omega, \mathbb{R}^2)$ be a sequence such that*

$$\sup_{\varepsilon>0} \mathrm{AG}_\varepsilon(u_\varepsilon) < +\infty \quad .$$

Then the sequence $(u_\varepsilon)_{\varepsilon>0}$ is relatively compact in L^1 for the strong convergence.

In [7], the study of the energy functionals AG_ε was motivated by its link with liquid crystals. Actually, one can also see (5.4.1) as a toy model for the following model for ferromagnetic samples.

5.4.2 Micromagnetics

The state of a ferromagnetic sample, represented by a bounded open set $\Omega \subset \mathbb{R}^3$, is characterized by a function $m = (m_1, m_2, m_3) : \Omega \to \mathbb{S}^2$, called magnetization, where $\mathbb{S}^2$ stands for the unit sphere in $\mathbb{R}^3$. In the theory of micromagnetics, the magnetization m represents a stable state of the following energy functional (considered here in the absence of external magnetic field):

$$F_\varepsilon(m) = d^2 \int_\Omega |\nabla m|^2 + \int_\Omega \phi(m) + \int_{\mathbb{R}^3} |H|^2, \tag{5.4.3}$$

where

- d is a small parameter called *exchange length*. The first term in (5.4.3), called *exchange energy*, penalizes the variations of m.
- The *anisotropy function* $\phi : \mathbb{S}^2 \to \mathbb{R}^+$ is some smooth function (such that $\inf \phi = 0$) which favors some directions for the magnetization, called easy axis, corresponding to the points where ϕ vanishes.
- $H \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ is some vector field induced by m, called *stray field* or *magnetostatic energy*. H is solution of the following problem:

$$\begin{cases} \nabla \times H = 0 & \text{in } \mathbb{R}^3, \\ \nabla \cdot H = -\nabla \cdot (m\mathbf{1}_\Omega) & \text{in } \mathbb{R}^3. \end{cases}$$

  In other words, $H = -\nabla u$ where $u : \mathbb{R}^3 \to \mathbb{R}$ is solution of $\Delta u = \nabla \cdot (m\mathbf{1}_\Omega)$ in $\mathbb{R}^3$. Thus $\int |H|^2 = \|\nabla \cdot (m\mathbf{1}_\Omega)\|^2_{H^{-1}(\Omega)}$ and the last term in (5.4.3) penalizes the divergence of $m\mathbf{1}_\Omega$.

We consider the following situation:

- $\Omega = \omega \times \mathbb{R}$ where $\omega \subset \mathbb{R}^2$ is a bounded open set and m does not depend on the third variable $x_3 \in \mathbb{R}$.
- We consider some regime where, after renormalization, (5.4.3) reads

$$F_\varepsilon(m) = \frac{\varepsilon}{2} \int_\omega |\nabla m|^2 + \frac{1}{2\varepsilon} \int_\omega \varphi(m) + \frac{1}{\beta} \int_{\mathbb{R}^2} |H|^2,$$

  where ε, $\beta = \beta(\varepsilon) > 0$ are some parameters depending on d, ϕ such that $\beta \ll \varepsilon$ and φ is the renormalized anisotropy.
- $\varphi : \mathbb{S}^2 \to \mathbb{R}$ favors directions in the plane $\{z_3 = 0\}$:

$$\begin{cases} \varphi(m) = 0 & \text{if } m \in \mathbb{S}^2 \cap \{m_3 = 0\}, \\ \varphi(m) > 0 & \text{otherwise.} \end{cases}$$

Since $\beta \ll \varepsilon$, $\int_{\mathbb{R}^2} |H|^2$ is the main term among the renormalized term $\int |\nabla m|^2$, $\int \varphi(m)$ and $\int |H|^2$. Since $\int |H|^2 = \|\nabla \cdot (m\mathbf{1}_\omega)\|^2_{H^{-1}(\mathbb{R}^2)}$ and $m(x) = m(x_1, x_2)$, a simplified model consists in considering a divergence constraint on $m' = (m_1, m_2)$:

$$E_\varepsilon(m) = \begin{cases} \dfrac{1}{2} \displaystyle\int_\omega \varepsilon |\nabla m|^2 + \frac{1}{\varepsilon}\varphi(m) & \text{if } \nabla \cdot m' = 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{5.4.4}$$

A finite energy sequence $(m_\varepsilon)_{\varepsilon > 0}$ for E_ε, is expected to converge to some divergence-free vector field $m : \omega \to \mathbb{S}^1$. Some experiments show that, at least for ε very small, the magnetization is smooth out of a thin layer (very close to a line) of size ε on which it changes very quickly between two values $m^\pm$ (see [35]). The microstructures formed by the magnetization into this layer can be more or less complex. In the simplest case, it is one-dimensional, i.e. it depends only on the normal (to the jump line) variable. However more complex structures can appear as cross-tie wall ([2], [57]) or zigzag-patterns ([38]) for example. If $\varphi(m) = |m_3|^\alpha$ with $0 < \alpha \leq 4$, only one-dimensional structures are expected. Thus, for these specific anisotropies, E_ε is expected to Γ-converge to the line-energy $\mathcal{I}_f$, where $f = c_\varphi^{1D}$ is defined in an analogous way as (5.3.3) for $\mathbb{S}^2$-valued maps. Here, $f(u^+, u^-, \nu) = c\, g(|u^+ - u^-|)$ where $g(t) = \frac{|u^+-u^-|^p}{p}$ with $p = 1 + \frac{\alpha}{2}$, is the primitive of $\sqrt{\varphi}$ vanishing at 0 and $c > 0$ is a constant. The case $\varphi(m) = |m_3|^2$ was studied by R. Ignat and B. Merlet in [36] in which a compactness result was proved and sharp lower bounds were found. However, the Γ-liminf property (5.1.3) was established only for limiting $1D$ configurations of the form $m(x) = \pm\nu^\perp$ for $\pm x \cdot \nu > 0$ with $\nu \in \mathbb{S}^1$ (see figure 5.1 in the case $\nu = e_3$).

Note that, when $\varphi(m) = m_3^4 = (1 - |m'|^2)^2$, E_ε corresponds to the Aviles-Giga AG_ε with an additional term, $\frac{\varepsilon}{2} \int |\nabla m_3|^2$: for all $m \in H^1(\omega, \mathbb{S}^2)$, one has

$$E_\varepsilon(m) = \mathrm{AG}_\varepsilon(m') + \frac{\varepsilon}{2} \int_\omega |\nabla m_3|^2 \quad .$$

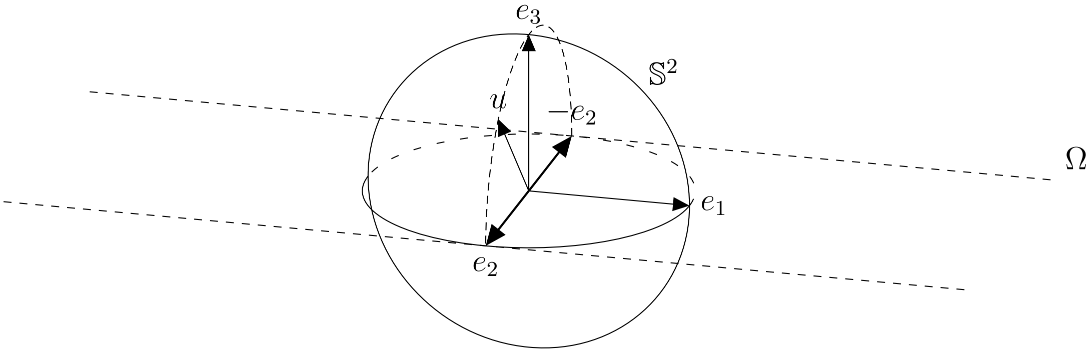

Figure 5.1 – Bloch wall for the maximal jump configuration: $\pm e_2$

Chapter 6

A necessary condition for lower semicontinuity of line energies

This chapter is based on a work (see [14]) in collaboration with Pierre Bochard (University of Paris-Sud).

We are interested in some energy functionals concentrated on the discontinuity lines of divergence-free 2D vector fields valued in the circle $\mathbb{S}^1$. This kind of energy has been introduced first by P. Aviles and Y. Giga in [6]. They show in particular that, with the cubic cost function $f(t) = t^3$, this energy is lower semicontinuous. In this paper, we construct a counter-example which excludes the lower semicontinuity of line energies for cost functions of the form t^p with $0 < p < 1$. We also show that, in this case, the viscosity solution corresponding to a certain convex domain is not a minimizer.

6.1 Introduction

6.1.1 Line energies

Let Ω be a Lipschitz domain in $\mathbb{R}^2$. We are interested in measurable vector fields $m\colon \Omega \to \mathbb{R}^2$ such that

$$|m| = 1 \text{ a.e. and } \nabla \cdot m = 0 \text{ on } \Omega, \tag{6.1.1}$$

where the second equation holds in the distributional sense. In the following, we will assume that m is of bounded variation so as to be able to define its *jump line*. So, we consider the set

$$A(\Omega) := \left\{ m \in BV(\Omega, \mathbb{R}^2) \,:\, |m| = 1 \text{ a.e. and } \nabla \cdot m = 0 \text{ on } \Omega \right\}.$$

Vector fields $m \in A(\Omega)$ are related to solutions of the *eikonal equation* in Ω. Let us define the set

$$S(\Omega) := \{\varphi \in \mathrm{Lip}(\Omega) \,:\, |\nabla\varphi| = 1 \text{ a.e. and } \nabla\varphi \in BV(\Omega)\}.$$

If Ω is simply connected, for all $m \in A(\Omega)$, there exists a scalar function $\varphi \in S(\Omega)$ such that

$$m(x) = (\nabla\varphi(x))^\perp \text{ a.e.},$$

where $(\nabla\varphi)^\perp = R\nabla\varphi$ stands for the image of $\nabla\varphi$ by the rotation R of angle $\pi/2$ centered at the origin in $\mathbb{R}^2$. Moreover, a function $\varphi \in \mathrm{Lip}(\Omega)$ satisfying $(\nabla\varphi)^\perp = m$ a.e. is unique up to a constant and is called stream function. We are now able to define line energies:

Definition 6.1.1. Let $f\colon [0,2] \to [0,+\infty]$ be a measurable scalar function. Let $m \in A(\Omega) \subset BV(\Omega,\mathbb{R}^2)$. Then, there exists a $\mathcal{H}^1$-rectifiable jump line $J(m)$ oriented by a unit normal vector ν_x such that m has traces $m_\pm(x) \in \mathbb{S}^1$ on each side of $J(m)$ for $\mathcal{H}^1$ a.e. $x \in J(m)$ (see [5] for more details). Then, the energy associated with the so called *jump cost* f is denoted by $\mathcal{I}_f$ and defined for $m \in A(\Omega)$ as follows:

$$\mathcal{I}_f(m) = \int_{J(m)} f(|m_+ - m_-|)\,\mathrm{d}\mathcal{H}^1(x).$$

Note that the divergence constraint on $m \in A(\Omega)$ implies that for *a.e.* $x \in J(m)$, $m_\pm(x) \in \mathbb{S}^1$ and ν_x satisfy the following condition (see figure 6.3):

$$m_+(x)\cdot\nu_x = m_-(x)\cdot\nu_x\ .$$

Then, in the orthogonal basis $(\nu_x,\nu_x^\perp)$, there exists some angle θ such that $m_\pm = (\cos\theta, \pm\sin\theta)$ and the jump size is defined as

$$t = |m_+ - m_-| = 2|\sin\theta|.$$

Similarly, $\mathcal{I}_f$ can be interpreted as a functional of the stream function on the set $S(\Omega)$: Writing $m = (\nabla\varphi)^\perp \in BV(\Omega,\mathbb{R}^2)$, then $\mathcal{I}_f(m) = \mathcal{J}_f(\varphi)$ where

$$\forall\varphi \in S(\Omega),\quad \mathcal{J}_f(\varphi) = \int_{J(\nabla\varphi)} f(|(\nabla\varphi)_+ - (\nabla\varphi)_-|)\,\mathrm{d}\mathcal{H}^1(x). \tag{6.1.2}$$

An interesting question is to find the minimizing structures of $\mathcal{I}_f$ if it exists. Remark that for this problem to be relevant, we have to consider a constraint on the boundary otherwise all constant functions are minimizers. A natural choice is to minimize $\mathcal{I}_f$ along all configurations m belonging to the set

$$A_0(\Omega) := \{m \in A(\Omega)\ :\ m\cdot n = 0 \text{ a.e. on } \partial\Omega\},$$

where n is the exterior unit normal vector of $\partial\Omega$. In terms of the stream function φ, this is equivalent to consider the set

$$S_0(\Omega) := \{\varphi \in S(\Omega)\ :\ \varphi = 0 \text{ on } \partial\Omega\}.$$

6.1.2 Lower semicontinuity, Viscosity solution

As explained above, some of the line energies $\mathcal{I}_f$ are conjectured to be the Γ-limit of functionals coming from micromagnetics in the space $X = L^1$. If that is the case, $\mathcal{I}_f$ has to satisfy the following lower semicontinuity property:

Definition 6.1.2. Let $F : X \to [0,+\infty]$ be a functional defined on some topological space X. F is said to be lower semicontinuous or l.s.c. if the following holds:

$$\forall (x_n)_{n\geq 0} \subset X, \quad x_n \underset{n\to+\infty}{\longrightarrow} x \implies F(x) \leq \liminf_{n\to\infty} F(x_n).$$

Since this property strongly depends on the topology of the space X, we have to specify the choice we make for the study of line energies $\mathcal{I}_f$.

First of all, due to the non-convex constraint $|m| = 1$, we need strong compactness in L^1. Moreover, since all the results of the previous part (compactness and Γ-liminf property) holds for the L^1 strong topology, it seems natural to consider the line energies $\mathcal{I}_f$ in the space $X = L^1$.

However, since Definition 6.1.1 uses the notion of trace of a function, another natural choice would be $X = BV$ endowed with the weak topology which is a very common choice for phase transition problems. Unfortunately, in the general case, the space BV is not adapted to our problem. Suppose $f(t) = t^p$ with $p > 1$ for instance. Then finite energy configurations m (i.e. $m_n \underset{n\to+\infty}{\longrightarrow} m$ in L^1 with $\mathcal{I}_f(m_n) \leq C < +\infty$) are not necessarily of bounded variation since the total variation of m around its jump line cannot be controlled by $\int_{J(m)} |m_+ - m_-|^p$ if $p > 1$ (see [4]). That is why we need a subspace of solutions of the problem (6.1.1) included in $L^1(\Omega)$ (and containing BV) because of the non-convex constraint $|m| = 1$ such that we are still able to define a jump line $J(m)$ and traces $m_\pm$. This is done in [27] where a regularity result is shown for solution of (6.1.1) with bounded "entropy production".

Note that if X and Y are two topological spaces such that Y is continuously embedded in X and $F : X \to [0,+\infty]$ is l.s.c. in X then the restriction of F to Y is l.s.c. in Y. In this paper, we only want to prove a necessary condition for functionals $\mathcal{I}_f$ to be l.s.c. We then prefer to restrict our analysis to BV functions (see remark 6.1.6).

In the case where $f(t) = t^p$ for some $p > 0$, only partial results are known. In [4], the following is conjectured:

Conjecture 6.1.3. Let $\overline{\mathcal{I}_f}$ be the relaxation of $\mathcal{I}_f$ (only defined on the space BV) in L^1:

$$\overline{\mathcal{I}_f}(m) = \operatorname{Inf}\left\{ \liminf_{n\to+\infty} \mathcal{I}_f(m_n) \ : \ m_n \in BV \text{ and } m_n \underset{n\to+\infty}{\longrightarrow} m \text{ in } L^1 \right\}.$$

If $f(t) = t^p$ with $1 \leq p \leq 3$ then $\overline{\mathcal{I}_f}$ is l.s.c. for the strong topology in L^1.

For $p > 3$, this conjecture is false (see [4]). The case $p = 3$ has been studied by P. Aviles and Y. Giga in [7]. More recently the case $p = 2$ has been proved by R. Ignat and B. Merlet in [37]. They also proved that Conjecture 6.1.3 holds true for $1 \leq p \leq 3$ if one restricts to configurations m such that the jump size is always lower than $\sqrt{2}$. Here we are interested in the open case $p < 1$.

We point out that line energies associated with the cost $f(t) = t^p$ with $1 \leq p \leq 3$ correspond exactly to the expected Γ-limits of functionals (5.4.4) when $\varphi(m) = |m_3|^\alpha$

with $0 < \alpha \leq 4$ where Bloch walls seem to be optimal. This is quite natural since when 2D structures, as cross tie wall or zigzag wall for instance, have less energy than Bloch walls, the Γ-limit of these functionals may be non lower semicontinuous. In the next part, we are going to give a 2D construction which gives some necessary condition on f for $\mathcal{I}_f$ to be l.s.c. This condition excludes cost functions of the form $f(t) = t^p$ with $p < 1$; the proof is based on a construction in the spirit of [2] and [57].

Theorem 6.1.4. *Let* $f\colon [0,2] \to [0,+\infty]$. *Let* Ω *be an open and bounded non empty subset of* $\mathbb{R}^2$. *Assume that* $\mathcal{I}_f$ *is lower semicontinuous in* $X = BV(\Omega,\mathbb{S}^1)$ *endowed with the weak topology. Then* f *is lower semicontinuous and we have*

$$\limsup_{t\to 0} \frac{f(t)}{t} \leq 2 \limsup_{t\to 2} f(t). \tag{6.1.3}$$

Remark 6.1.5. The fact that the lower semicontinuity of $\mathcal{I}_f$ implies the lower semicontinuity of f has already been proved in [37]. The main new point here is the condition (6.1.3).

Remark 6.1.6. Theorem 6.1.4 is stronger than an equivalent formulation in which BV is replaced by some Banach space X such that BV is continuously embedded in X and where $\mathcal{I}_f$ is replaced by its relaxation in X.

As we will see, the lower semicontinuity of functionals $\mathcal{I}_f$ is closely related to the following question: Is the viscosity solution a minimizer of $\mathcal{I}_f$? In [37], the authors address the following conjecture

Conjecture 6.1.7. Assume that $\overline{\mathcal{I}_f}$ is l.s.c. in L^1 and that Ω is convex. Then $(\nabla\varphi_0)^\perp$ is a global minimizer of $\mathcal{I}_f$ where $\varphi_0(x) = \operatorname{dist}(x,\partial\Omega)$.

For a regular domain Ω the distance function $\varphi_0(x) = \operatorname{dist}(x,\partial\Omega)$ belongs to $S_0(\Omega)$ and $(\nabla\varphi_0)^\perp$ is the viscosity solution of the problem (6.1.1). In particular, if Ω is convex, φ_0 is concave and $-D^2\varphi_0$ is a positive vectorial radon measure. In [4], the authors give a microstructure which shows that the viscosity solution is not a minimizer if $f(t) = t^p$ with $p > 3$. As explained below, we are going to give a structure with lower energy than the viscosity solution for $p < 1$.

Proposition 6.1.8. *Let* $f : [0,2] \to [0,+\infty]$. *There exists a convex domain* Ω *such that the following holds. Let* $\varphi_0 \in S_0(\Omega)$ *be the distance function* $\varphi_0(x) = \operatorname{dist}(x,\partial\Omega)$. *Assume that* φ_0 *is a minimizer of* $\mathcal{J}_f$ *defined by* (6.1.2). *Then f satisfies* (6.1.3).

Corollary 6.1.9. *There exists a convex domain* Ω *such that the viscosity solution is not a minimizer of* $\mathcal{I}_f$ *if* $f(t) = t^p$ *with* $p \in [0,1[$.

We finish by the following result which gives a partial answer to Question 1 (see (5.3.5) page 102):

Proposition 6.1.10. *Let* $W : \mathbb{R}^2 \to \mathbb{R}^+$ *be some* $\mathcal{C}^1$ *potential such that* $W(z) = 0 \Leftrightarrow z \in \mathbb{S}^1$. *Let us consider the energy* E_ε *for* $\varepsilon > 0$ *defined in* (5.1.2), c_W^{per} *in* (5.3.2) *and* c_W^{1D} *in* (5.3.3). *Assume that*

$$c_W^{1D}(m^+,m^-) = c_W^{per}(m^+,m^-) = f(|m^+ - m^-|) \textit{ for all } m^\pm \in \mathbb{S}^1 \quad ,$$

where $f : [0,2] \to [0,+\infty]$ *is some function. Then* $f = c_W^{1D}$ *satisfies* (6.1.3).

6.2 Construction of a competitor of the viscosity solution

In order to obtain the inequality (6.1.3), we have to construct a domain Ω on which the jump size $t = |m_+ - m_-|$ of the viscosity solution along its singular set is very small. Then, we find a competitor whose jump size t is close to the maximal possible value $t = 2$. In other words, we want to substitute small jumps by large ones.

We will use the polar coordinates (r, θ), $r \geq 0$, $\theta \in [-\pi, \pi]$ and we will identify $\mathbb{R}^2$ and $\mathbb{C}$ with the usual bijection. Let D be the unit disk and $\mathcal{C}$ be its boundary.

Let θ_0 be a fixed angle in $]0, \pi/2[$ and define the two points $A = e^{i\theta_0}$ and $A' = e^{-i\theta_0}$ on the circle $\mathcal{C}$. Define also T_A (resp. $T_{A'}$) the tangent to the circle $\mathcal{C}$ at the point A (resp. A'). We consider the domain Ω delimited by the large arc $\{e^{i\theta} \ : \ |\theta| > \theta_0\}$, T_A and $T_{A'}$ (see figure 6.1). In other words Ω is the interior of the convex envelope of $\mathcal{C} \cup \{B\}$ where $B = T_A \cap T_{A'}$. Define also $\Omega_0 = \Omega \cap \{|\theta| < \theta_0 \text{ and } r > 0\}$ and $\Gamma = \partial\Omega \cap \partial\Omega_0 = [AB] \cup [A'B]$.

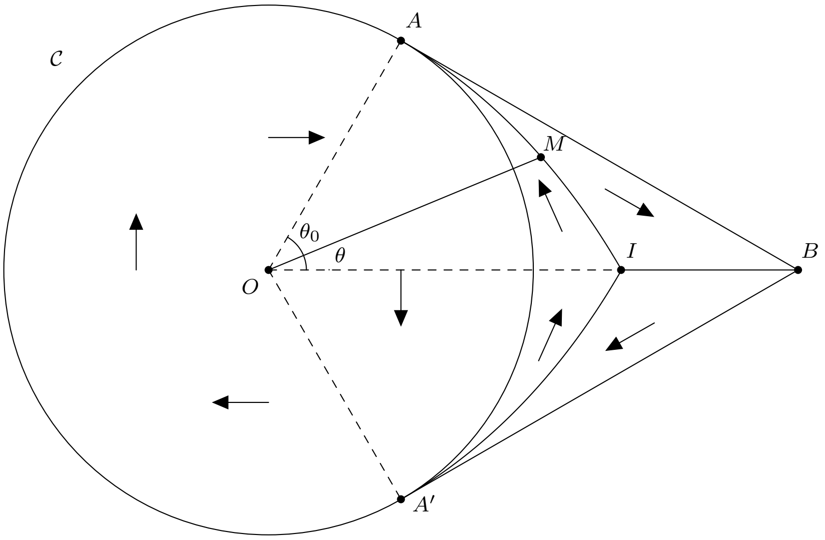

Figure 6.1 – The domain Ω and the microstructure m

We now consider two solutions φ_0 and φ in $S_0(\Omega)$ of the eikonal equation vanishing on the boundary:

— φ_0 is the usual distance function: $\forall x \in \Omega,\ \varphi_0(x) = \text{dist}(x, \partial\Omega)$.

— φ defined by: $\forall x \in \Omega,\ \varphi(x) = \text{dist}(x, \partial\Omega \cup \mathcal{C})$.

We also denote by $m_0 = (\nabla\varphi_0)^\perp$ and $m = (\nabla\varphi)^\perp$ the corresponding solutions of (6.1.1). Then $m_0, m \in A_0(\Omega)$.

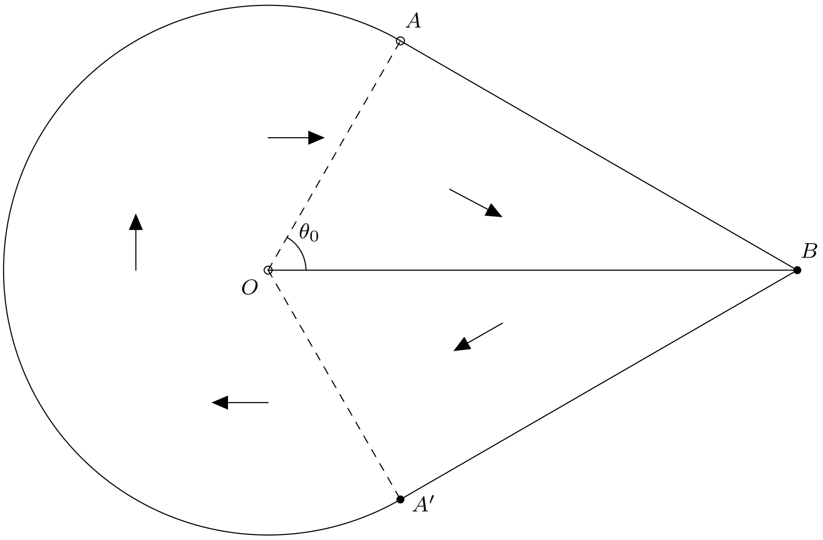

Figure 6.2 – Viscosity solution m_0 on Ω

We now compute $\mathcal{I}_f(m_0)$ and $\mathcal{I}_f(m)$ in order to prove that the function φ has lower energy than φ_0 if $f(t) = t^p$ with $p < 1$.

Heuristic: The idea is that a small jump along a fixed length is replaced by big jumps on a small length : This will reduce the energy for subadditive power costs (i.e. $f(t) = t^p$ with $p < 1$) which favor "small jumps". Let us give more details.

For a small angle $\theta_0 > 0$, m_0 only presents small jumps: m_0 is $\mathcal{C}^1$ out of segment $[OB]$ on which the jump size is $|m_0^+ - m_0^-| =: t_0 = 2\sin(\theta_0)$.

On the contrary, m only presents "big" jumps: i.e. jumps whose size is close to 2. The singular set of m consists in 3 different lines : $[IB]$ whose length is equivalent to θ_0^2 and the two curves $\mathcal{C} \setminus \mathcal{C}_{\theta_0}$ and γ_{θ_0} (defined below) on which the jump size tends to 2 and the length of these lines is equivalent to $2\theta_0$.

As a result, the energy of m_0 is close to $f(2\sin\theta_0)$ while the energy of m is close to $4\theta_0 \times f(2)$. A necessary condition for m_0 to minimize $\mathcal{I}_f$ is then (see Proposition 6.1.8)

$$\limsup_{t\to 0} f(t)/t \leq 2f(2).$$

This excludes subadditive power costs. In the sequel, we are going to make precise computations so as to get more informations about the critical angle θ_0.

Energy of m_0: The jump line of m_0 is the segment $[OB]$ and the traces of m_0 on each side of this line are given by $m_{0,\pm} = -e^{i(\pi/2\pm\theta_0)}$. In particular,

$$\mathcal{I}_f(m_0) = f(2\sin\theta_0)|OB| = \frac{f(2\sin\theta_0)}{\cos\theta_0}.$$

Energy of m: The jump line of m is the union of the 3 curves:

— $\mathcal{C}_{\theta_0} = \{e^{i\theta} \, : \, |\theta| < \theta_0\}$.

— $\gamma_{\theta_0} := \{z \in \Omega_0 \, : \, d(z, \mathcal{C}_{\theta_0}) = d(z,\Gamma)\} = \{z = re^{i\theta} \, : \, |\theta| < \theta_0, \, d(z,\mathcal{C}) = d(z, \partial\Omega)\}$.

— The segment $[IB]$ where $I = \gamma_{\theta_0} \cap [OB]$.

First, let us find a polar equation for the curve γ_{θ_0}: Given $z = re^{i\theta}$ such that $|\theta| < \theta_0$ and $r > 1$ we have $d(z, \mathcal{C}_{\theta_0}) = r - 1$, it remains to compute $\lambda := d(z, \Gamma)$.

Since Ω is symmetric with respect to the axis (OB), one can restrict to the case $M = r\, e^{i\theta}$ with $0 < \theta < \theta_0$. So $\lambda := d(z,\Gamma) = |z - P|$ where P is the orthogonal projection of $M = re^{i\theta}$ on the segment $[AB]$: P should satisfy $\overrightarrow{MP} = \lambda \overrightarrow{OA} = \lambda\, e^{i\theta_0}$ and $\overrightarrow{MP} \cdot \overrightarrow{AP} = 0$. We then compute

$$\begin{aligned}\overrightarrow{MP} \cdot \overrightarrow{AP} =& \overrightarrow{MP} \cdot [\overrightarrow{AO} + \overrightarrow{OM} + \overrightarrow{MP}]\\ =&\Re\{\lambda\, e^{-i\theta_0}\, (-e^{i\theta_0} + r\, e^{i\theta} + \lambda\, e^{i\theta_0})\}\\ =&\lambda[-1 + r\cos(\theta_0 - \theta) + \lambda].\end{aligned}$$

Since $\overrightarrow{MP} \cdot \overrightarrow{AP} = 0$, this implies $\lambda = MP = 1 - r\cos(\theta_0 - \theta)$. Then we have $z \in \gamma_{\theta_0}$ if and only if $r - 1 = 1 - r\cos(\theta_0 - \theta)$ and the polar equation of the curve γ_{θ_0} is given by

$$r(\theta) = \frac{2}{1 + \cos(\theta_0 - |\theta|)} \; ; \; -\theta_0 < \theta < \theta_0 \quad .$$

Now, we can compute the energy of m along the curve γ_{θ_0}:

— $\mathrm{d}\gamma(\theta) = \sqrt{r(\theta)^2 + r'(\theta)^2}\, \mathrm{d}\theta$ where we find $r'(\theta) = \dfrac{-2\sin(\theta_0 - \theta)}{(1 + \cos(\theta_0 - \theta))^2}$. Introducing the notation $\alpha = \theta_0 - \theta$, we obtain

$$\mathrm{d}\gamma(\theta) = 2\frac{\sqrt{(1+\cos\alpha)^2 + \sin^2\alpha}}{(1+\cos\alpha)^2}\,\mathrm{d}\theta = 2\frac{\sqrt{2(1+\cos\alpha)}}{(1+\cos\alpha)^2}\,\mathrm{d}\theta = \frac{4\cos(\alpha/2)}{(2\cos^2(\alpha/2))^2}\,\mathrm{d}\theta.$$

So $\mathrm{d}\gamma$ reads

$$\mathrm{d}\gamma(\theta) = \cos^{-3}(\alpha/2)\,\mathrm{d}\theta.$$

— The size of the jump at the point $\gamma(\theta)$ is given by

$$t(\theta) = |m_+ - m_-| = |e^{i(\theta_0 + \pi/2)} + e^{i(\theta + \pi/2)}| = |e^{i(\theta_0 - \theta)} + 1|.$$

Using once again the notation $\alpha = \theta_0 - \theta$, this gives

$$t(\theta) = \sqrt{(\cos\alpha + 1)^2 + \sin^2\alpha} = \sqrt{2(1+\cos\alpha)} = 2\cos(\alpha/2).$$

— We conclude that the energy of m induced by the jump line γ_{θ_0} is given by

$$\mathcal{I}^1_f(m) = \int_{-\theta_0}^{\theta_0} \frac{f[2\cos(\alpha/2)]}{\cos^3(\alpha/2)}\,\mathrm{d}\alpha.$$

The energy concentrated on the arc $\mathcal{C}_{\theta_0}$ is

$$\mathcal{I}_f^2(m) = f(2)\,\mathcal{H}^1(\mathcal{C}_{\theta_0}) = 2\theta_0\, f(2).$$

Finally, we compute the energy on the line $[IB]$:

$$\mathcal{I}_f^3(m) = f(2\sin\theta_0)|IB|.$$

If the distance function is a minimizer of $\mathcal{I}_f$ we should have

$$\mathcal{I}_f(m) - \mathcal{I}_f(m_0) \geq 0.$$

Now, the preceding equations yields

$$\begin{aligned}\mathcal{I}_f(m) - \mathcal{I}_f(m_0) =& \mathcal{I}_f^1(m) + \mathcal{I}_f^2(m) + \mathcal{I}_f^3(m) - \mathcal{I}_f(m_0)\\ =& \int_{-\theta_0}^{\theta_0} \frac{f[2\cos(\alpha/2)]}{\cos^3(\alpha/2)}\,\mathrm{d}\alpha + 2\theta_0\, f(2) + (|IB| - |OB|)\, f(2\sin\theta_0).\end{aligned}$$

Since $|IB| - |OB| = -|OI| = -r(0) = -\dfrac{1}{\cos^2(\theta_0/2)}$, this gives

$$\mathcal{I}_f(m) - \mathcal{I}_f(m_0) = \int_{-\theta_0}^{\theta_0} \frac{f[2\cos(\alpha/2)]}{\cos^3(\alpha/2)}\,\mathrm{d}\alpha + 2\theta_0 f(2) - \frac{f(2\sin\theta_0)}{\cos^2(\theta_0/2)}.$$

Hence, if m_0 is a minimizer of $\mathcal{I}_f$, the following condition should be satisfied:

$$\begin{aligned}\frac{f(2\sin\theta_0)}{2\sin\theta_0} &\leq \frac{\theta_0\cos^2(\theta_0/2)}{\sin\theta_0}\left[\frac{1}{\theta_0}\int_0^{\theta_0} \frac{f[2\cos(\alpha/2)]}{\cos^3(\alpha/2)}\,\mathrm{d}\alpha + f(2)\right]\\ &\leq \frac{\theta_0}{\sin\theta_0\cos(\theta_0/2)} \times 2\ \sup\{f(t)\ :\ 2\cos(\theta_0/2) \leq t \leq 2\}.\end{aligned}$$

Finally, taking the lim sup for $\theta_0 \to 0$ in the preceding equation leads to (6.1.3):

$$\limsup_{t\to 0} \frac{f(t)}{t} \leq 2\limsup_{t\to 2} f(t).$$

This proves Proposition 6.1.8 and corollary 6.1.9 follows from the fact that the preceding inequality holds false for $f(t) = t^p$ with $p < 1$. Note that in this case, we get something more precise than Proposition 6.1.8:

Proposition 6.2.1. *Assume that $f(t) = t^p$ with $p < 1$. There exists $\theta_0 \in]0, \pi/2[$ only depending on p such that for all $\theta \in]-\theta_0, \theta_0[$, the viscosity solution is not a minimizer of $\mathcal{I}_f$ on Ω_θ where Ω_θ is the convex set constructed in the previous part (θ being the angle $(\overrightarrow{OB}, \overrightarrow{OA})$).*

6.3 Lower semicontinuity of line energies, proof of Theorem 6.1.4

The fact that if $\mathcal{I}_f$ is l.s.c. then f is l.s.c. can be found in [37] (Proposition 1). In this section we prove that (6.1.3) is a necessary condition for $\mathcal{I}_f$ to be lower semicontinuous with respect to the weak convergence in BV on bounded open subsets of $\mathbb{R}^2$. The proof is based on an elementary homogenization principal.

The key is to use the construction $m \in A(\Omega)$ depending on θ_0 of the first part by restriction to Ω_0 (See figure 6.3.). The $1D$ transition defined by (6.3.1) corresponds to the viscosity solution m_0 of the previous part. Given a small parameter $\epsilon > 0$, it will costs less energy to substitute the $1D$ transition around its jump line by the microstructure m rescaled at the level ϵ (see figure 6.4).

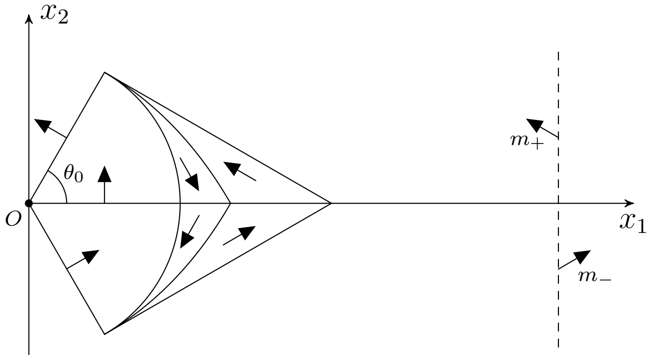

Figure 6.3 – The vector field m on the left and the 1D-transition m_0 on the right

We are going to prove Theorem 6.1.4 when $\Omega = (0,1) \times (-1,1)$. The general case will follow easily. Fix $\theta_0 \in]0, \pi/2[$ and define the $1D$ transition m_0 for a.e. $x_1 \in (0,1)$ and $x_2 \in \mathbb{R}$ by

$$m_0(x_1, x_2) = m_\pm := (\mp \sin\theta_0, \cos\theta_0) \text{ if } \pm x_2 > 0. \tag{6.3.1}$$

Then, let us consider the vector field $m = m_{\theta_0}$ of the preceding section restricted to Ω_0 and define the rescaled and extended vector field $\tilde{m}$ for $x_1 \in (0,1)$ and $x_2 \in \mathbb{R}$:

$$\tilde{m}(x_1, x_2) = \begin{cases} -m\left((\cos\theta_0)^{-1}\, x_1, (\cos\theta_0)^{-1}\, x_2\right) & \text{if } ((\cos\theta_0)^{-1}\, x_1, (\cos\theta_0)^{-1}\, x_2) \in \Omega_0, \\ m_0(x_1, x_2) & \text{otherwise.} \end{cases}$$

Note that $\tilde{m}$ belongs to $A(\Omega)$ and is continuous up to the boundary, $\tilde{m} \in \mathcal{C}\left(\overline{\Omega}_0\right)$. Then, let n be a positive integer and define $m_n \in A(\Omega)$ by aligning n times the vector field $\tilde{m}$ (see figure 6.4). More precisely, for $0 \le i < n$ and $x = (x_1, x_2) \in \Omega$ such that $i/n \le x_1 < (i+1)/n$, define

$$m_n(x_1, x_2) = \tilde{m}(n\, x_1 - i, n\, x_2).$$

We have $m_n(x_1, x_2) = m_0(x_1, x_2)$ for $|x_2| > 1/n$ and for all $x \in \Omega$, $|m_n(x)| = 1$. Consequently, $(m_n)_{n>0}$ converge to m_0 in $L^1(\Omega)$. Moreover, $|m_n|_{BV(\Omega)} = |\tilde{m}|_{BV(\Omega)}$ so that $(m_n)_{n>0}$ is bounded in $BV(\Omega)$ and weakly converges to m_0.

Since m_n is obtained by scaling a fixed structure, it is easy to see that $\mathcal{I}_f(m_n)$ is constant. Indeed, $\mathcal{I}_f(m_n) = n \times 1/n\ \mathcal{I}_f(\tilde{m}) = \mathcal{I}_f(\tilde{m})$. That is why we obtain the

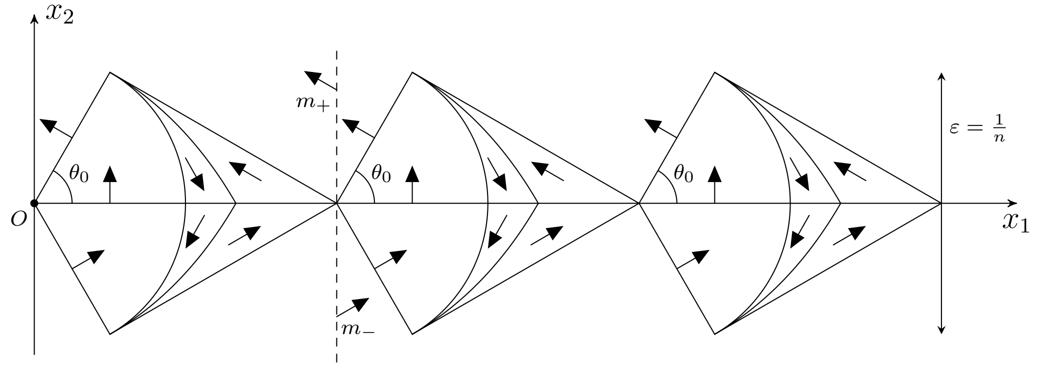

Figure 6.4 – The microstructure m_n

following condition: assuming $\mathcal{I}_f$ is *l.s.c.*,

$$\mathcal{I}_f(m_0) = f(2\,\sin\theta_0) \leq \liminf_{n\to\infty} \mathcal{I}_f(m_n) = \mathcal{I}_f(\tilde{m}).$$

In other words, the viscosity solution costs less energy than the construction m_{θ_0} of the preceding part. For this reason, we obtain exactly the same necessary condition (6.1.3) and this ends the proof when $\Omega = (0,1)\times(-1,1)$.

In the general case, let D be an horizontal line such that $D \cap \Omega \neq \emptyset$. Up to a translation of Ω, we can assume that $0 \in D$, i.e. $D = \{x \in \mathbb{R}^2 \ : \ x_2 = 0\}$. Fix $\theta_0 \in]0,\pi/2[$ and define $\tilde{m}_0 \in \mathrm{BV}(\Omega,\mathbb{S}^1)$ by:

$$\text{for a.e. } x \in \Omega,\ \tilde{m}_0(x_1,x_2) := m_\pm = (\mp\sin\theta_0, \cos\theta_0) \quad \text{if} \quad \pm x_2 > 0 \quad . \tag{6.3.2}$$

Let $q \in \mathbb{R}$ and $r > 0$ and define $\Omega_{q,r} = (q, q+r)\times(-r,r)$. n being fixed, we obtain a microstructure $m_n^{q,r}$ defined on $\Omega_{q,r}$ by rescaling and translating our microstructure m_n defined on $(0,1)\times(-1,1)$. More precisely, for a.e. $x \in \Omega_{q,r}$,

$$m_n^{q,r}(x_1,x_2) := m_n\left(\frac{x_1-q}{r}, \frac{x_2}{r}\right).$$

Now, there exists two sequences $(q_i)_{i\in\mathbb{N}}$, $(r_i)_{i\in\mathbb{N}}$ such that

$$D \cap \Omega \subset \bigcup_{i=0}^{\infty} \overline{\Omega_{q_i,r_i}} \quad ,$$

where for $i \neq j$, $\Omega_{q_i,r_i} \cap \Omega_{q_j,r_j} = \emptyset$. We can now define the microstructure $\tilde{m}_n$ for a.e. x belonging to the whole domain Ω by

$$\tilde{m}_n(x) = \begin{cases} m_n^{q_i,r_i}(x) & \text{if } x \in \Omega_{q_i,r_i} \text{ for some } i \in \mathbb{N}, \\ \tilde{m}_0(x) & \text{otherwise,} \end{cases}$$

where $\tilde{m}_0$ has been defined in (6.3.2). It is clear that $\tilde{m}_n \xrightarrow[n\to\infty]{} \tilde{m}_0$ in $L^1(\Omega)$ and in the weak BV sense. Furthemore,

$$\mathcal{I}_f(\tilde{m}_0) = \mathrm{length}(\Omega\cap D)\ \mathcal{I}_f(m_0) \leq \liminf_{n\to\infty} \mathcal{I}_f(\tilde{m}_n) = \mathrm{length}(\Omega\cap D)\ \liminf_{n\to\infty} \mathcal{I}_f(m_n).$$

Since $\Omega \cap D \neq \emptyset$ and Ω is bounded, one has $+\infty > \mathrm{length}(\Omega\cap D) > 0$ which imply the thesis.

6.4 Optimality of the $1D$ profile, proof of Proposition 6.1.10

Let $f : [0,2] \to [0,+\infty]$ be the function defined by $f(|m^+ - m^-|) := c_W^{per}(m^+, m^-) = c_W^{1D}(m^+, m^-)$ for $m^\pm \in \mathbb{S}^1$. Assume that f does not satisfy the condition (6.1.3). Let us consider the domain $\Omega = (0,1) \times (-1,1)$, the $1D$ transition m_0 defined in (6.3.1) and the $2D$ structure $\tilde{m}$ defined below. By construction, as (6.1.3) is not satisfied, one has

$$\mathcal{I}_f(\tilde{m}) < \mathcal{I}_f(m_0) = f(|m^+ - m^-|) = c_W^{1D}(u^+, u^-).$$

Moreover, thanks to Theorem 5.3.2, there exists a sequence $(m_\varepsilon)_{\varepsilon>0} \subset \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}^2)$ such that $E_\varepsilon(m_\varepsilon, \Omega) \xrightarrow[\varepsilon\to 0]{} \mathcal{I}_f(\tilde{m}, \Omega)$. Now, by definition of c_W^{per}, one has

$$c_W^{per}(m^+, m^-) \leq \liminf_{\varepsilon \to 0} E_\varepsilon(m_\varepsilon, \Omega) = \mathcal{I}_f(\tilde{m}, \Omega) < c_W^{1D}(m^+, m^-).$$

Chapter 7

A De Giorgi conjecture for divergence-free vector fields

This chapter is based on a work in collaboration with my advisor, Radu Ignat.

In its classical framework, the De Giorgi conjecture claims that every entire solution of the scalar elliptic semilinear equation $\Delta u - W'(u) = 0$ which satisfies some monotonicity property only depends on a single variable (see [25, 3]). Under some strong assumptions on the potential, one can prove that a similar result holds true for divergence-free vector fields. In this chapter, we prove the symmetry for global minimizers of some Aviles-Giga type energy in periodic strips with divergence constraint. Our main assumption is that the potential W is the square of a harmonic function or a solution of the wave equation. Our proof is based on the entropy method introduced by P. Aviles and Y. Giga in order to study some simplified Ginzburg-Landau models.

7.1 Introduction

7.1.1 Main question

Let $d \geq 1$ be the dimension and let $\Omega = \mathbb{R} \times \mathbb{T}^{d-1}$ be an infinite cylinder, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $x' = (x_2, \dots, x_n)$. We consider the functional defined by

$$E(u) = \begin{cases} \frac{1}{2} \int_\Omega |\nabla u|^2 + W(u)\,\mathrm{d}x & \text{if } u \in \dot{H}^1(\Omega) \text{ and } \nabla \cdot u = 0, \\ +\infty & \text{otherwise,} \end{cases} \tag{7.1.1}$$

where $W : \mathbb{R}^d \to \mathbb{R}^+$ is a nonnegative potential and $\dot{H}^1(\Omega)$ is the subspace of $H^1_{loc}(\Omega, \mathbb{R}^d)$ defined by

$$\dot{H}^1(\Omega) = \left\{ u \in H^1_{loc}(\Omega, \mathbb{R}^d) \ : \ \nabla u \in L^2(\Omega, \mathbb{R}^{d\times d}) \right\}.$$

In order to make the problem of minimizing the energy $E(u)$ non trivial, we will impose a boundary condition on u. One possibility would be to impose pointwise convergence of u at infinity: for a.e. $x_2 \in \mathbb{T}^{d-1}$,

$$u(\pm\infty, x_2) := \lim_{x_1 \to \pm\infty} u(x_1, x_2) = u^\pm =: (u_1^\pm, \overline{u}^\pm) = (a, \overline{u}^\pm), \tag{7.1.2}$$

where $u^\pm \in \mathbb{R}^d$ are two wells, i.e. $W(u^\pm) = 0$ and, because of the divergence constraint, $a := u_1^+ = u_1^- \in \mathbb{R}$. Actually, we will only need the following weaker condition: we impose that the x'-average of u is a continuous function in $x_1 \in \mathbb{R}$ having the following limit at infinity,

$$\lim_{x_1 \to \pm\infty} \int_{\mathbb{T}^{d-1}} u(x_1, x')\,\mathrm{d}x' = u^\pm. \tag{7.1.3}$$

Alternatively, we consider the weaker condition

$$u(\lambda\cdot,\cdot) \xrightarrow{\lambda\uparrow\infty} \begin{cases} u^- & \text{if } x_1 < 0 \\ u^+ & \text{if } x_1 > 0 \end{cases} \quad \text{in } L^1_{loc}(\Omega).$$

Our aim is to analyze the following De Giorgi type problem:

Question: *Under which conditions on the potential W, is it true that every global minimizer u of E over the set of divergence-free vector fields satisfying the boundary condition* (7.1.2) *is one-dimensional, i.e. $u = u(x_1)$, and unique up to a translation in x_1-direction* ?

7.1.2 Analysis of the one-dimensional profile

We assume that $W : \mathbb{R}^d \to \mathbb{R}^+$ is nonnegative and continuous. The *singular set*, or *well set*, is denoted by S and defined by $S := \{z \in \mathbb{R}^d \,:\, W(z) = 0\}$.

We look for optimal vector fields $u : \Omega \to \mathbb{R}^d$ for the energy E, only depending on the first variable x_1, which are divergence-free and satisfies $u(\pm\infty,\cdot) = u^\pm$. Let us write $u^\pm = (a, \overline{u}^\pm)$ and $u : \Omega \to \mathbb{R}^d$ as

$$u(x) = u(x_1, x') =: \varphi(x_1),$$

where $\varphi : \mathbb{R} \to \mathbb{R}^d$ satisfies the boundary condition $\varphi(\pm\infty) = u^\pm$. Since $\nabla\cdot u = 0 = \partial_1\varphi_1$, the first component of φ is constant, say $\varphi_1 \equiv a \in \mathbb{R}$ and φ reads

$$\varphi(x) = (a, \gamma(x_1)) \quad \text{for } x \in \Omega, \tag{7.1.4}$$

where $\gamma : \mathbb{R} \to \mathbb{R}^{d-1}$ satisfies $\gamma(\pm\infty) = \overline{u}^\pm$. Thus, $E(u) = E^{1D}(\gamma)$ where the one-dimensional energy E^{1D} is defined by

$$E^{1D}(\gamma) = \frac{1}{2}\int_{\mathbb{R}} |\gamma'(t)|^2 + W(a, \gamma(t))\,\mathrm{d}t \quad . \tag{7.1.5}$$

We are going to investigate the existence of a global minimizer for E^{1D} under the constraint $\gamma(\pm\infty) = \overline{u}^\pm$. Note that any global minimizer satisfies the Euler-Lagrange equation,

$$2\gamma''(t) = D_{x'}W(a, \gamma(t)),$$

in the distributional sense. In particular, multiplying the preceding equation by γ' and integrating provides the equipartition of the energy:

$$|\gamma'(t)|^2 = W(a, \gamma(t)).$$

For, any curve $\gamma : \mathbb{R} \to \mathbb{R}^{d-1}$ satisfying the equipartition identity, the following Young inequality is sharp:

$$E^{1D}(\gamma) \geq \int_{\mathbb{R}} \sqrt{W(a,\gamma(t))}|\gamma'(t)|\,\mathrm{d}t \quad . \tag{7.1.6}$$

Note that the last integral, invariant by monotone reparametrization, represents the length of γ in the hyperspace $\mathbb{R}^d_a := \{z \in \mathbb{R}^d \ : \ z_1 = a\}$ endowed with the Riemannian metric $g_W(z) = W(z)g$, where g is the usual Euclidean metric on $\mathbb{R}^d_a$. Thus every global minimizer γ of E^{1D} connecting $\overline{u}^-$ to $\overline{u}^+$ lies on a shortest geodesic from $\overline{u}^-$ to $\overline{u}^+$ in $(\mathbb{R}^d_a, g_W)$. This metric becomes singular at the points z where $W(z) = 0$. However, at least if the singular set S is discrete, it induces a distance on the space $\mathbb{R}^d_a$: for all $\overline{u}^\pm \in \mathbb{R}^d_a$,

$$\begin{aligned} d_{w,a}(\overline{u}^-, \overline{u}^+) := \inf \Bigg\{ \int_{-1}^{1} w(a,\gamma(s))|\gamma'(s)|\,\mathrm{d}s \ : \ \gamma \in \mathrm{Lip}([-1,1], \mathbb{R}^d_a), \\ \gamma(\pm 1) = \overline{u}^\pm \Bigg\}, \end{aligned} \tag{7.1.7}$$

where $w := \sqrt{W}$. If $u^\pm := (a, \overline{u}^\pm)$ are two wells, i.e. $W(u^\pm) = 0$, and γ is global minimizer of E^{1D} under the constraint $\gamma(\pm\infty) = \overline{u}^\pm$ then the preceding considerations easily yield $E^{1D}(\gamma) = d_{w,a}(\overline{u}^-, \overline{u}^+)$. In general, a global minimizer could not exist (see Proposition 7.1.2) but the infimum of E^{1D} always coincides with $d_{w,a}(\overline{u}^-, \overline{u}^+)$ (see Proposition A.3.2).

Two dimensional case In dimension $d = 2$, the situation is very simple since $\mathbb{R}^d_a$ is of dimension 1 (see figure 7.1). In particular it is clear that, in order to compute the infimum of E^{1D}, one can assume that γ is monotone. Then, the change of variables $y = \gamma(t)$ in (7.1.6) yields

$$\inf \left\{ E^{1D}(\gamma) \ : \ \gamma \in H^1_{loc}(\mathbb{R}),\, \gamma(\pm\infty) = u_2^\pm \right\} = \int_{u_2^-}^{u_2^+} \sqrt{W(a,y)}\,\mathrm{d}y \quad . \tag{7.1.8}$$

The existence of the one-dimensional profile is given by the following proposition.

Proposition 7.1.1. *Assume that $W(a,y) > 0$ for all $y \in (u_2^-, u_2^+)$. Then, there exists a unique (up to translation) global minimizer γ_{1D} of E^{1D} under the constraint $\gamma_{1D}(\pm\infty) = u_2^\pm$. If $\pm(u_2^+ - u_2^-) \geq 0$, then γ_{1D} is the unique (up to translation) solution of the ode*

$$\gamma'_{1D}(t) = \pm\sqrt{W(a, \gamma_{1D}(t))} \tag{7.1.9}$$

such that $\gamma_{1D}(\pm\infty) = u_2^\pm$. Moreover $E(\gamma_{1D}) = \int_{u_2^-}^{u_2^+} \sqrt{W(a,\gamma)}\,\mathrm{d}\gamma$. Conversely, if $\sqrt{W}$ is locally Lipschitz and if there exists $y \in (u_2^-, u_2^+)$ such that $W(a,y) = 0$, then the infimum of $E^{1D}(\gamma)$ under the constraint $\gamma(\pm\infty) = u_2^\pm$ is not achieved.

Proof. If $u_2^+ = u_2^-$, the proposition is trivial. One can assume that $u_2^- \neq u_2^+$, say $u_2^- < u_2^+$. First assume that $W(a,y) > 0$ for $y \in (u_2^-, u_2^+)$. Thanks to the Peano-Arzelà, since $\sqrt{W}$ is continuous, there exists at least one maximal solution $\gamma : I \subset \mathbb{R} \to [u_2^-, u_2^+]$

of the ode (7.1.9) (where the sign ± is a +) such that $\gamma(0) = \frac{u_2^- + u_2^+}{2}$ (see figure 7.2). Since $\sqrt{W}$ is bounded on $[u_2^-, u_2^+]$ and since $W(u^\pm) = 0$, by maximality, γ is global, i.e. defined on $I = \mathbb{R}$. Moreover $\gamma(\pm\infty) = u_2^\pm$. Indeed, since γ is non-decreasing, $\gamma(\pm\infty)$ exist and are stationary points of (7.1.9). Thus, one has

$$E^{1D}(\gamma) = \int_{\mathbb{R}} \sqrt{W(a,\gamma(t))}\, \gamma'(t)\, \mathrm{d}t = \inf\left\{ E^{1D}(\gamma) \;:\; \gamma \in \dot{H}^1(\mathbb{R}),\, \gamma(\pm\infty) = u_2^\pm \right\}.$$

Conversely, assume that $\sqrt{W}$ is locally Lipschitz and that there exists $y \in (u_2^-, u_2^+)$ such that $W(a, y) = 0$. Assume furthermore that γ is global minimizer of E^{1D} under the constraint $\gamma(\pm\infty) = u_2^\pm$. Then, by optimality, γ is non-decreasing. Indeed if there was two instants $u < v$ such that $\gamma(u) > \gamma(v)$, by continuity, one could find $w > v$ such that $\gamma(w) = \gamma(u)$. Then one could strictly reduce the energy of γ replacing it by the constant value $\gamma_0 := \gamma(u) = \gamma(v)$ on the interval (u, w) which is a contradiction with the optimality of γ. Now, as $\gamma' \geq 0$, using the equirepartition identity, we deduce that γ is a solution of the ode (7.1.9) (where the sign "±" is a "+") and, up to translation, one can assume that $\gamma(0) = y$. Yet, since $\sqrt{W}$ is locally Lipschitz, solutions to the ode (7.1.9) with the initial condition $\gamma(0) = y \in (u_2^-, u_2^+)$ are unique which is a contradiction with the fact that $\gamma_0 \equiv y \in (u_2^-, u_2^+)$ is a stationary solution of (7.1.9).

□

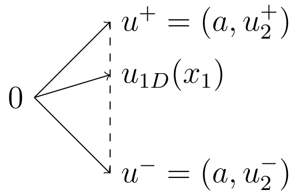

Figure 7.1 – Geodesic from u^- to u^+

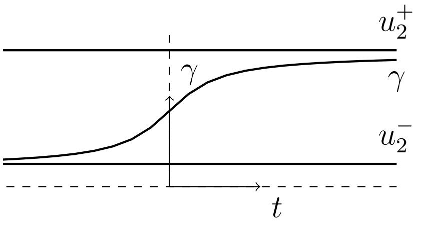

Figure 7.2 – One-dimensional profile

Higher dimension In dimension greater than 3, the problem of the existence of a one-dimensional minimizer is more tricky. We are going to give reasonable sufficient conditions on the potential W which insure the existence of a global minimizer for E^{1D}. Since the space $(\mathbb{R}^d_a, g_W)$ is not a smooth manifold due to the singularities, when $S \neq \emptyset$, it is natural to establish the existence of minimizers for E^{1D} in the general framework of length spaces. Indeed, even if $S \neq \emptyset$, at least if S is discrete, $(\mathbb{R}^d_a, d_{w,a})$ is a length space. This study is included in the annex of this thesis. Here, we are content with giving the following proposition, without proof, which is sufficient for our purpose:

Proposition 7.1.2. *Fix $d \geq 1$ the dimension and let $W : \mathbb{R}^d \to \mathbb{R}^+$ be a potential. Let us fix two wells $u^\pm = (a, \overline{u}^\pm) \in \mathbb{R}^d$. Assume that $W_a(\cdot) := W(a, \cdot)$, defined on $\mathbb{R}^d_a := \{(a, u) \;:\; u \in \mathbb{R}^{d-1}\}$, satisfies the following conditions:*

1. *$d_* := \inf\{|u - v| \;:\; u, v \in \mathbb{R}^d_a,\ W(u) = W(v) = 0$ and $u \neq v\} > 0$.*

2. $\omega(\varepsilon) := \inf\{W_a(u) \ : \ u \in \mathbb{R}^d_a,\ d(u,S) > \varepsilon\} > 0$ *for all* $\varepsilon > 0$.
3. *There exists a geodesic* $\gamma_0 : \overline{u}^- \to \overline{u}^+$ *in the space* $(\mathbb{R}^d_a, d_w)$ *such that* $\mathrm{Im}(\gamma_0) \cap S = \{\overline{u}^\pm\}$ *where* $w := \sqrt{W_a}$.

Then the following one-dimensional minimization problem has a solution:

$$\inf\left\{E^{1D}(\gamma) := \frac{1}{2}\int_{\mathbb{R}} |\gamma'(t)|^2 + W(a,\gamma(t))\,\mathrm{d}t \ : \right.$$
$$\left. \gamma \in H^1_{loc}(\mathbb{R},\mathbb{R}^{d-1}),\ \gamma(\pm\infty) = \overline{u}^\pm\right\}.$$

Proof. We refer to the annex for a proof. Proposition 7.1.2 is the same as Corollary A.3.8 page 178. ☐

7.2 One-dimensional symmetry: proof of the results in $2D$

In this section, we are going to prove the one-dimensional symmetry for global minimizers, under boundary conditions, of the energy E defined over divergence-free vector fields $u \in \dot{H}^1(\Omega, \mathbb{R}^2) = \{u \in H^1_{loc}(\Omega,\mathbb{R}^2) \ : \ \nabla u \in L^1\}$ (see Theorem 7.2.13 below):

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + W(u),$$

where $\Omega = \mathbb{R}\times\mathbb{T}$ and $W \in \mathcal{C}^1(\mathbb{R}^2,\mathbb{R}^+)$ is a nonnegative potential satisfying the following growth condition:

$$\exists p > 0,\ C > 0,\ \forall z \in \mathbb{R}^2,\ W(z) \leq C(1+|z|^p). \tag{7.2.1}$$

In addition to (7.2.1), we will restrict to potentials satisfying a strong assumption for which we are able to deduce rigidity results. Namely, we will impose that W is the square of some entire solution of either the Laplace equation or the wave equation: there exists $w : \mathbb{R}^2 \to \mathbb{R}$ such that for all $z \in \mathbb{R}^2$, $W(z) = w^2(z)$ and

$$\partial_{11} w \pm \partial_{22} w = 0. \tag{7.2.2}$$

Remark 7.2.1. Note that this property, which will be our main assumption, is very strong. For instance, if w is a harmonic function with polynomial growth at infinity (see (7.2.1)), then it is a polynomial. The fact that we need such a strong assumption on the potential makes a significant difference with the De Giorgi conjecture in the classical scalar framework where weaker conditions are required (see [3]). However this is not surprising since, in our context, the one-dimensional symmetry for global minimizer is a very strong property which is not true for generic potentials. For instance, the Ginzburg-Landau theory for micromagnetics provides several examples where the energy can be strictly reduced by $2D$-structures. These phenomena arise from the particular structure of the magnetization in magnetics domains, eventually submitted to an external magnetic field, and have been observed in several experiments (see [35]) or constructed theoretically (see [38, 2, 4, 14]).

Remark 7.2.2. The advantage of the Laplace operator compared to the wave operator is that it is rotation invariant. Consequently, if w is harmonic, then our main symmetry result, Theorem 7.2.13, will also hold in an infinite cylinder in any direction $\nu \in \mathbb{S}^1$. If W is a multi-well potential (with at least three non aligned wells) this invariance property allows to deduce a rigidity result for the transition between any two wells.

As before, in addition to the divergence constraint, we impose the following boundary condition:

$$\lim_{x_1 \to \pm\infty} \int_{\mathbb{T}} u(x_1, x_2)\ \mathrm{d}x_2 = u^{\pm}, \tag{7.2.3}$$

where $u^{\pm}$ are two wells, i.e. $W(u^{\pm}) = 0$ and, because of the divergence constraint, $a := u_1^+ = u_1^- \in \mathbb{R}$. We will need the following lemma which provides stronger convergence of u to the prescribed values $u^{\pm}$ at the boundary:

Lemma 7.2.3. *Assume that $u \in H^1_{loc}(\Omega)$ satisfies* (7.2.3)*, then $u(x_1, x_2)$ converges up to a subsequence to $u^{\pm}$ when $x_1 \to \pm\infty$ for the uniform convergence: there exists two sequences R_n^+ and R_n^- such that $R_n^{\pm} \to \pm\infty$ and*

$$|u(R_n^{\pm}, \cdot) - u^{\pm}|_{L^\infty(\mathbb{T})} \underset{n\to\infty}{\longrightarrow} 0,$$

where $u(R, .)$ stands for the trace of the function u at $x_1 = R \in \mathbb{R}$.

Proof. We use the following Sobolev inequality: for all $u \in H^1_{loc}(\Omega)$,

$$\int_{\mathbb{R}} \left| u(x_1, .) - \int_{\mathbb{T}} u(x_1, t) dt \right|_{\infty}^2 dx_1 \leq \int_{\Omega} |\nabla u|^2 \quad , \tag{7.2.4}$$

where $|.|_\infty$ is the L^∞ norm. This is a consequence of the classical one-dimensional Sobolev inequality: $\left|u(x_1, .) - \int_{\mathbb{T}} u(x_1, t)dt\right|_\infty \leq \int_{\mathbb{T}} |\partial_2 u(x_1, x_2)| dx_2$ for almost every $x_1 \in \mathbb{R}$ and the Jensen inequality. In view of (7.2.4), we know that the function defined by $f(x_1) = \left|u(x_1, .) - \int_{\mathbb{T}} u(x_1, t)dt\right|_\infty^2$ is integrable on $\mathbb{R}$ and we can find two sequences $R_n^{\pm} \to \infty$ such that $f(R_n^+)$ and $f(R_n^-)$ converge to 0 as $n \to \infty$ which, together with (7.2.3) gives the result. □

A fundamental observation which will be very useful is Proposition 7.2.4 below, valid in every dimension. In particular, it states that the Dirichlet energy, that is the squared L^2-norm of ∇u, can be replaced by the squared L^2 norm of $\nabla \times u$ where $\nabla\times$ stands for the curl operator: $\nabla \times u = -\partial_2 u_1 + \partial_1 u_2$. For technical reasons, if the potential W is the square of a harmonic function (resp. a solution of the wave equation) then it is easier to deal with $\int |\nabla \times u|^2$ (resp. $\int |\partial_1 u_2 + \partial_2 u_1|^2$) instead of the Dirichlet energy in the estimates.

Proposition 7.2.4. *Let $d \geq 1$ be the dimension and $u = (u_1, \dots, u_d) \in \dot{H}^1(\Omega, \mathbb{R}^d)$ be a divergence-free vector field satisfying the boundary condition* (7.1.3)*. Then, one has*

$$\int_{\Omega} |\nabla u|^2 = \int_{\Omega} \sum_{i<j} |\partial_i u_j - \partial_j u_i|^2 = \int_{\Omega} \sum_{i<j} |\partial_i u_j + \partial_j u_i|^2 + 2 \int_{\Omega} \sum_{i} |\partial_i u_i|^2, \tag{7.2.5}$$

where i, j are integers in $\{1, \dots, d\}$. In other words, if P^+ (resp. P^-) denotes the projection of $\mathbb{R}^{d\times d}$ on the subspace composed of symmetric (resp. antisymmetric) matrices, that is $P^\pm U = \frac{1}{2}(U \pm U^T)$ for $U \in \mathbb{R}^{d\times d}$, then one has $\|P^-\nabla u\|^2_{L^2} = \|P^+\nabla u\|^2_{L^2}$ and so

$$\int_\Omega |\nabla u|^2 = \int_\Omega |P^-\nabla u|^2 + \int_\Omega |P^+\nabla u|^2 = 2\int_\Omega |P^-\nabla u|^2 = 2\int_\Omega |P^+\nabla u|^2.$$

Proof. Since $u \in \dot{H}^1(\Omega)$, up to convolution with a smooth kernel, one can assume that $u \in \mathcal{C}^\infty(\Omega)$. Then, we compute

$$\sum_{i<j} |\partial_i u_j \pm \partial_j u_i|^2 = \sum_{i\neq j} |\partial_i u_j|^2 \pm 2\sum_{i<j} \partial_i u_j \partial_j u_i = \sum_{i\neq j} \left(|\partial_i u_j|^2 \pm \partial_i u_j \partial_j u_i\right).$$

Now, since $\nabla \cdot u = 0$, one has $0 = |\nabla \cdot u|^2 = |\sum_i \partial_i u_i|^2 = \sum_{i,j} \partial_i u_i \partial_j u_j$ so that

$$\sum_{i<j} |\partial_i u_j - \partial_j u_i|^2 = |\nabla u|^2 - \sum_{i,j} \partial_i u_j \partial_j u_i = |\nabla u|^2 - \sum_{i,j} \left(\partial_i u_j \partial_j u_i - \partial_i u_i \partial_j u_j\right).$$

In order to prove the first identity in (7.2.5), we have to prove that integrating the last term of the preceding equation, we obtain 0. Let us use the notation $b_{ij} = \partial_i u_j \partial_j u_i - \partial_i u_i \partial_j u_j \in L^1(\Omega)$. For all $i, j \in \{1, \dots, d\}$, one has $b_{ij} = b_{ji}$ and $b_{ii} = 0$. Moreover, if $i, j \neq 1$, since x_i and x_j lie on the torus $\mathbb{T}$ which has no boundary, integrating by parts twice yields $\int_\Omega b_{ij} = 0$. Thus, $\int_\Omega \sum_{ij} b_{ij} = 2\sum_{j\neq 1}\int_\Omega b_{1j}$. Let us define $\Omega_R := [-R, R] \times \mathbb{T}^{d-1}$ for every $R > 0$. Integrating by parts on Ω_R twice and using the divergence constraint yield

$$\begin{aligned}
\sum_{j\neq 1}\int_{\Omega_R} b_{1j} &= \sum_{j\neq 1}\int_{\Omega_R} \partial_1 u_j \partial_j u_1 - \partial_1 u_1 \partial_j u_j \\
&= -\sum_{j\neq 1}\int_{\Omega_R} \partial_{1j} u_j u_1 - \partial_1 u_1 \partial_j u_j \\
&= -\sum_{j\neq 1}\int_{\mathbb{T}^{d-1}} u_1(R, x')\partial_j u_j(R, x') - u_1(-R, x')\partial_j u_j(-R, x')\,\mathrm{d}x' \\
&= \int_{\mathbb{T}^{d-1}} u_1(R, x')\partial_1 u_1(R, x') - u_1(-R, x')\partial_1 u_1(-R, x')\,\mathrm{d}x' \\
&\leq \|u_1\|_{L^2(\partial\Omega_R)}\|\partial_1 u_1\|_{L^2(\partial\Omega_R)}.
\end{aligned}$$

Now, since $\nabla u \in L^2(\Omega)$, there exists a sequence $(R_n)_{n\geq 1}$ converging to $+\infty$ such that $\|\nabla u\|_{L^2(\partial\Omega_{R_n})} \underset{n\to\infty}{\longrightarrow} 0$. Thanks to the Poincaré-Wirtinger inequality, we have also

$$\|u_1\|_{L^2(\partial\Omega_{R_n})} \leq \left|\int_{\mathbb{T}^{d-1}} u_1(R, x')\,\mathrm{d}x'\right| + \left|\int_{\mathbb{T}^{d-1}} u_1(-R, x')\,\mathrm{d}x'\right| + \|\nabla_{x'} u_1\|_{L^2(\partial\Omega_{R_n})},$$

which is bounded because of the boundary condition (7.1.3). Hence, we have $\|u_1\|_{L^2(\partial\Omega_{R_n})}\|\partial_1 u_1\|_{L^2(\partial\Omega_{R_n})} \underset{n\to\infty}{\longrightarrow} 0$ and we deduce, in the limit when n goes to ∞, that $\sum_{j\neq 1}\int_\Omega b_{1j} = 0$ as claimed. For the second equality of (7.2.5), one has

$$\begin{aligned}
\sum_{i<j} |\partial_i u_j + \partial_j u_i|^2 &= |\nabla u|^2 + \sum_{i,j} \partial_i u_j \partial_j u_i - 2\sum_i |\partial_i u_i|^2 \\
&= |\nabla u|^2 - 2\sum_i |\partial_i u_i|^2 + \sum_{i,j} \left(\partial_i u_j \partial_j u_i - \partial_i u_i \partial_j u_j\right)
\end{aligned} \tag{7.2.6}$$

and, once again, (7.2.5) follows from the fact that integrating the last term of the preceding equality, we obtain a boundary term which vanishes. □

The main tool for the study of global minimizers of the energy E (under both the divergence constraint and (7.2.3)) is the entropy method which has reminiscence in the works of Aviles-Giga, Jin-Kohn and which has been formalized in [36]. Originally, it has been used in the Aviles-Giga model, that is $W(u) = (1-|u|^2)^2$, to show that the one-dimensional transition layer is optimal (see [39]). Here, we are going to prove that the one-dimensional transition layer is actually unique up to a translation. Note that, in this case, the potential W is the square of a solution of the wave equation on $\mathbb{R}^2$ which will be our main hypothesis on W. Then, we will extend this condition on the potential to the new situation where $W(z) = w^2(z)$ for some harmonic function w.

For the convenience of the reader, we recall the entropy method in a simplified version, sufficient for our problem. The main idea is to estimate the energy density from below by some expression of the form $\nabla \cdot \{\Phi(u)\}$ for some $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ in such a way that this estimate is sharp for the one-dimensional profile which is our candidate for the global minimization problem. Namely, we look for locally Lipschitz maps $\Phi = (\Phi_1, \Phi_2) \in \mathrm{Lip}_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ and $\alpha \in \mathrm{Lip}_{loc}(\mathbb{R}^2)$ such that for every $u \in \mathcal{C}^\infty(\Omega, \mathbb{R}^2)$ which is bounded on Ω, there holds,

$$\nabla \cdot \{\Phi(u)\} + \alpha(u) \nabla \cdot u \leq \frac{1}{2}|\partial_1 u_2 \mp \partial_2 u_1|^2 + \frac{1}{2} W(u) \qquad \text{a.e. in } \Omega, \tag{7.2.7}$$

where the sign "$\mp$" in the first term of the RHS is chosen according to the sign in (7.2.2), namely "$-$" for the harmonic case and "$+$" for the wave equation. Note that, because of Proposition 7.2.4, if u is divergence-free, then $\int_\Omega \nabla \cdot \{\Phi(u)\}$ is less or equal than the energy $E(u)$. As we will see later, (7.2.7) implies that $\nabla\Phi$ can be controlled by $\sqrt{W}$ (see (7.3.10)). In this part we will look for entropies such that $|\nabla\Phi|^2$ satisfies the same growth condition as W, that is (7.2.1):

$$\exists p > 0,\ C > 0,\ \forall z \in \mathbb{R}^2,\ |\nabla\Phi(z)|^2 \leq C(1 + |z|^p). \tag{7.2.8}$$

Together with (7.2.7), the saturation condition with respect to (7.1.2) is fundamental: namely, we will impose that

$$\Phi_1(u^+) - \Phi_1(u^-) = \int_{u_2^-}^{u_2^+} \sqrt{W(a,t)}\, dt \left(= \int_{u_2^-}^{u_2^+} \partial_2 \Phi_1(a,t) dt \quad \text{for smooth } \Phi \right), \tag{7.2.9}$$

which exactly means that inequality (7.2.7) integrated on the whole domain is sharp for the one-dimensional transition layer. Indeed, for a smooth vector field u which is divergence-free and satisfies (7.2.3), the integration of (7.2.7) on Ω rewrites $\Phi_1(u^+) - \Phi_1(u^-) \leq E(u)$. For this reason, we get the following proposition

Proposition 7.2.5. *Assume that there exists $\Phi \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R}^2)$ and $\alpha \in \mathcal{C}^0(\mathbb{R}^2, \mathbb{R})$ satisfying* (7.2.7) *and* (7.2.9)*. Then the infimum of the energy under the divergence constraint and the boundary condition* (7.2.3) *is equal to the infimum of the one-dimensional energy* (7.1.5) *under the same boundary condition:*

$$\inf\left\{ E(u) \ :\ u \in \dot{H}^1(\Omega) \ s.t.\ (7.2.3) \ and\ \nabla \cdot u = 0 \right\} = \int_{u_2^-}^{u_2^+} \sqrt{W(a,t)}\, \mathrm{d}t \quad .$$

A-priori other global minimizers might exist in that context. We will see later some situations where the one-dimensional transition layer (7.1.4) is the unique global minimizer. Before proving this proposition, we will need the following Lemma which is a direct consequence of the polynomial growth condition on the potential W, (7.2.1):

Lemma 7.2.6. *Assume that $u \in H^1_{loc}(\Omega)$ has finite energy, i.e. $\nabla\cdot u = 0$ and $E(u) < \infty$. Then, there exists a divergence-free sequence of smooth and bounded functions $(u_k)_{k\geq 0} \subset \mathcal{C}^\infty(\Omega)\cap L^\infty(\Omega)$ converging to u in $H^1_{loc}(\Omega)$ such that u_k satisfies our boundary condition* (7.2.3) *for all $k \geq 0$ and*

$$e(u_k) \underset{k\to\infty}{\longrightarrow} e(u) \quad in \quad L^1_{loc} \quad ,$$

where for all $v \in H^1_{loc}$, $e(v)$ stands for the energy density, that is $e(v) = \frac{1}{2}(|\nabla v|^2 + W(v))$.

Sometimes, when this property holds, the set of smooth and bounded functions is said to be dense in energy in the admissible set $\{u \in H^1_{loc}(\Omega) \ : \ \nabla\cdot u = 0 \text{ and } E(u) < \infty\}$. A situation where this property does not hold was pointed out for the first time by M.A. Lavrentiev in 1927 (see [44]). In particular, an example of functional whose infimum over smooth functions is strictly greater than the infimum over all admissible functions was given. This phenomenon is usually called "Lavrentiev gap".

Proof of Lemma 7.2.6. Let us fix $u \in H^1_{loc}(\Omega)$ satisfying all assumptions of Lemma 7.2.6. Let us fix some smooth kernel $\rho \in \mathcal{C}^\infty_0(\mathbb{R})$ such that $\rho \geq 0$, $\int_{\mathbb{R}} \rho = 1$ and define ρ_k by $\rho_k(t) = k\rho(kt)$ for $t \in \mathbb{R}$. Then we introduce the regularization of u defined by $u_k = \rho_k^{\otimes 2} * u$ for $k \geq 1$ where $\rho^{\otimes 2}(x) = \rho_k(x_1)\rho_k(x_2)$ for all $x \in \Omega$. Then u_k converges to u in $H^1_{loc}(\Omega)$, $u_k \in \mathcal{C}^\infty(\Omega)$ and $\nabla \cdot u_k = 0$. Moreover, u_k still satisfied our boundary condition (7.2.3). Indeed, the Fubini theorem yields

$$\begin{aligned}\int_{\mathbb{T}} u_k(x_1,x_2)\,\mathrm{d}x_2 &= \int_{\mathbb{R}} \rho_k(y_1) \int_{\mathbb{T}} \rho_k(y_2) \int_{\mathbb{T}} u(x_1-y_1, x_2-y_2)\,\mathrm{d}x_2\,\mathrm{d}y_2\,\mathrm{d}y_1 \\ &= \rho_k * \left\{ \int_{\mathbb{T}} u(\cdot, x_2)\,\mathrm{d}x_2 \right\}(x_1).\end{aligned}$$

Then, the convergence of $\int_{\mathbb{T}} u(x_1,x_2)\,\mathrm{d}x_2$ for $x_1 \to \pm\infty$ implies that of $\int_{\mathbb{T}} u_k(x_1,x_2)\,\mathrm{d}x_2$. It remains to check that u_k is bounded. To this aim, let us decompose u_k as follows:

$$u_k = \rho_k * \overline{u} + \rho_k^{\otimes 2} * (u - \overline{u}), \tag{7.2.10}$$

where

$$\overline{u}(x) = \overline{u}(x_1) = \int_{\mathbb{T}} u(x_1,x_2)\,\mathrm{d}x_2 \quad .$$

Since $u \in H^1_{loc}(\Omega)$, $\overline{u}$ is continuous and, because of (7.2.3), this implies that $\overline{u}$ is bounded on $\mathbb{R}$. In particular, $\|\rho_k * \overline{u}\|_{L^\infty} \leq \|\rho_k\|_{L^1}\|\overline{u}\|_{L^\infty} < \infty$. For the second term in the right hand side of (7.2.10), note that (7.2.4) implies in particular that $u - \overline{u} \in L^2(\Omega)$ as well as $\rho_k^{\otimes 2}$. By the Young inequality, we deduce that $\rho_k^{\otimes 2} * (u - \overline{u})$ is bounded.

It remains to prove the convergence of $e(u_k)$ in L^1_{loc}. We already know that, by construction, ∇u_k converges to ∇u in L^2. We have to prove the convergence of $W(u_k)$

in L^1_{loc}. Fix $R>0$ and define $\Omega_R=[-R,R]\times\mathbb{T}$. We want to prove that $W(u_k)$ converges to $W(u)$ in $L^1(\Omega_R)$. Since u_k converges to u in H^1_{loc} and so a.e. , we know that $W(u_k)$ converges almost everywhere to $W(u)$ in Ω. By the Vitali convergence theorem, it is enough to show that the sequence $(W(u_k))_{k\geq 1}$ is uniformly integrable in Ω_R.

Because of the polynomial growth condition (7.2.1), we can find $C\geq 1$ and $p\geq 1$ such that $\forall z\in\mathbb{R}^2$, $W(z)\leq C(1+|z|^p)$. Let us fix $\varepsilon>0$. By a classical Sobolev imbedding, we deduce that $u\in H^1(\Omega_R)\subset L^p(\Omega_R)$. In particular there exists $\delta>0$ such that for all subset $A\subset\Omega_R$ whose Lebesgue measure $|A|$ satisfies $|A|\leq\delta$, we have $\int_A|u|^p\leq\varepsilon$. Now, for all $A\subset\Omega_R$ such that $|A|\leq\delta$, we can estimate

$$\int_A W(u_k)\leq C\left(\int_A 1+|u_k|^p\right)\leq C\left(|A|+\varepsilon\right)\leq C(\delta+\varepsilon),$$

where we have used the inequality $\int_A|u_k|^p\leq\varepsilon$. Indeed, by the Jensen inequality and the Fubini theorem, one has

$$\begin{aligned}\int_A|u_k|^p&\leq\int_A\left|\int_\Omega\rho_k^{\otimes 2}(y)u(x-y)\,\mathrm{d}y\right|^p\mathrm{d}x\leq\int_\Omega\int_A|u(x-y)|^p\,\mathrm{d}x\,\rho_k^{\otimes 2}(y)\,\mathrm{d}y\\&\leq\int_{A-y}|u(x)|^p\,\mathrm{d}x\leq\varepsilon\end{aligned}$$

since $|A-y|=|A|\leq\delta$. This conclude the proof of the uniform integrability of $(W(u_k))_k$ on Ω_R. ☐

Proof of Proposition 7.2.5. We are going to check rigorously the few ideas presented above. We have already seen that the infimum of the one-dimensional energy reads $E_*^{1D}:=\int_{u_2^-}^{u_2^+}\sqrt{W(a,t)}\,dt<\infty$ (see (7.1.8)). In order to prove that $E(u)\geq E_*^{1D}$ for every divergence-free vector field u satisfying the boundary condition (7.1.2), it is enough to assume that u is of finite energy $E(u)<\infty$. First assume that u is smooth and bounded. Then an integration by part and (7.2.7) imply that for all $R>0$,

$$\begin{aligned}\int_{\Omega_R}\nabla\cdot\{\Phi(u)\}\,\mathrm{d}x&=\int_{\mathbb{T}}\{\Phi_1(u(R,x_2))-\Phi_1(u(-R,x_2))\}dx_2\\&\leq\frac{1}{2}\int_{\Omega_R}(\partial_1u_2\pm\partial_2u_1)^2+W(u)\,\mathrm{d}x\quad,\end{aligned}\tag{7.2.11}$$

where $\Omega_R=[-R,R]\times\mathbb{T}$. Then, applying (7.2.11) to $R=R_n$ where the sequence R_n is given by Lemma 7.2.3, and passing to the limit when $n\to\infty$ yields

$$\int_{\mathbb{T}}\{\Phi_1(u^+)-\Phi_1(u^-)\}\,\mathrm{d}x_2=\int_{u_2^-}^{u_2^+}\sqrt{W(a,t)}\,\mathrm{d}t\leq E(u),$$

where we have used the saturation condition (7.2.9) and Proposition 7.2.4. This finishes the proof in the smooth case.

If u is not smooth or not bounded, we have to be careful since Φ is not globally Lipschitz continuous and $\Phi(u)$ is not necessarily in H^1 anymore. We apply Lemma

7.2.6 which provides a sequence of smooth and bounded functions u_k satisfying all the properties stated in Lemma 7.2.6. In particular u_k converges in H^1 and, up to extraction, one cas assume that u_k and ∇u_k converge a.e. Then we can apply the same estimate as above with u_k instead of u to get, for all $R > 0$,

$$\int_{\Omega_R} \nabla \cdot \{\Phi(u_k)\}\,\mathrm{d}x \leq \frac{1}{2}\int_{\Omega_R} (\partial_1 u_{k2} \pm \partial_2 u_{1k})^2 + W(u_k)\,\mathrm{d}x \quad . \tag{7.2.12}$$

Note that, since $\Phi \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R}^2)$, $\nabla[\Phi(u_k)] = \nabla\Phi(u_k)\nabla u_k$ converges a.e. to $\nabla[\Phi(u)] = \nabla\Phi(u)\nabla u$ as $k \to \infty$. Moreover, the growth condition, (7.2.8), and the Young inequality yields

$$|\nabla[\Phi(u_k)]| \leq |\nabla\Phi(u_k)|\ |\nabla u_k| \leq C\ (1 + |u_k|^p + |\nabla u_k|^2),$$

for some constant $C > 0$. In particular, the same proof as that of Lemma 7.2.6, shows that $(\nabla[\Phi(u_k)])_k$ is uniformly integrable on Ω_R. From the Vitaly convergence theorem, we deduce that $\nabla[\Phi(u_k)]$ converges in L^1_{loc} to $\nabla[\Phi(u)] = \nabla\Phi(u)\nabla u \in L^1_{loc}(\Omega)$, that is

$$\Phi(u_k) \underset{k\to\infty}{\longrightarrow} \Phi(u) \quad \text{in} \quad W^{1,1}_{loc}(\Omega) \quad .$$

In particular, $\nabla \cdot \{\Phi(u_k)\} \underset{k\to\infty}{\longrightarrow} \nabla \cdot \{\Phi(u)\}$ in $L^1_{loc}(\Omega)$ and passing to the limit in (7.2.12) yields (7.2.11). Indeed, since $\Phi(u) \in W^{1,1}$, one can integrate by parts. Thus, we can do all the computations we did in the smooth case and conclude, as before, that $E(u)$ is greater than the infimum of the one-dimensional energy. □

Now, we want to investigate the potentials W such that there exists an entropy $\Phi \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R}^2)$ satisfying (7.2.7) and (7.2.9). It is easy to see that (7.2.7) together with (7.2.9) implies that $\partial_2\Phi_1$ is determined on the geodesic (u^-, u^+):

Proposition 7.2.7. *Assume that $\Phi \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R}^2)$ satisfies (7.2.7). Then, for all $z \in \mathbb{R}^2$ such that $W(z) \neq 0$, one has*

$$\partial_2\Phi_1(z) \leq \sqrt{W(z)}. \tag{7.2.13}$$

Assume, in addition, that Φ satisfies the saturation condition, (7.2.9), and that $W(z) > 0$ on (u^-, u^+). Then one has

$$\forall z \in (u^-, u^+),\ \partial_2\Phi_1(z) = \sqrt{W(z)}. \tag{7.2.14}$$

Proof. For (7.2.13), we apply the inequality (7.2.7) at the point $x = 0$ to the vector field u defined by $u(x_1, x_2) = z + \chi(x_1)\sqrt{W(z)}\,x_1 e_2$ for some fixed $z \in \mathbb{R}^2$, where $\chi \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^+)$ is a cut-off function such that $\chi \equiv 1$ on a neighborhood of $x_1 = 0$. Note that, even if (7.2.7) only holds a.e., we can apply this inequality for the specific point $x = 0$ since each term in this inequality is continuous. We obtain $\partial_2\Phi_1(z)\sqrt{W(z)} \leq W(z)$ and (7.2.13) follows from the fact that $W(z) \neq 0$. The fact that (7.2.13) and (7.2.9) implies (7.2.14) is quite straightforward. □

Remark 7.2.8. Applying (7.2.7) to $u(x_1, x_2) = z + \chi(x_1)\sqrt{W(z)}\,x_2 e_1$, we also get $\partial_1\Phi_2(z) \leq \sqrt{W(z)}$ provided $W(z) \neq 0$ (see also [36]).

In order to get inequality (7.2.7) it is natural to impose that

$$\nabla\cdot\{\Phi(u)\}+\alpha(u)\nabla\cdot u=\sigma(u)\sqrt{W(u)}(\partial_1 u_2\mp\partial_2 u_1)\quad\text{a.e. in }\Omega$$

for all smooth and bounded vector field u, where σ is a measurable function from $\mathbb{R}^2$ to $\{\pm 1\}$ and the sign "$\mp$" is a "$-$" for the harmonic case and a "$+$" for the wave equation. σ and $w(z)$, defined in (7.2.2), are in fact related by the equation $w(z)=\sigma(z)\sqrt{W(z)}$. Since $\nabla\cdot\{\Phi(u)\}+\alpha(u)\nabla\cdot u=\mathrm{Tr}(\{\nabla\Phi(u)+\alpha(u)Id\}\nabla u)$, where Tr stands for the trace operator and Id is the 2×2 identity matrix, this preceding equation is equivalent to the following punctual condition on the differential of Φ:

$$\forall z\in\mathbb{R}^2,\ \nabla\Phi(z)+\alpha(z)Id=\begin{pmatrix}0 & \sigma(z)\sqrt{W(z)}\\ \mp\sigma(z)\sqrt{W(z)} & 0\end{pmatrix},$$

where the choice of the sign $\mp$ is the same sign that in (7.2.2). Note that, in the case where $\mp=-$, by Cauchy-Riemann, this condition implies that Φ is holomorphic on the whole space $\mathbb{R}^2$. That's why we will impose, in this case, that the potential W is the square of a harmonic function $w:\mathbb{R}^2\to\mathbb{R}$: $\forall z\in\mathbb{R}^2,\ W(z)=w^2(z)$. In this situation, due to the classical maximum principal, the set $\{W=0\}=\{w=0\}:=\{z\in\mathbb{R}^2 : w(z)=0\}$ cannot be a discrete set or a closed curve like $\mathbb{S}^1$ for instance. In fact, if w is not constant, then $\{w=0\}$ is a union (possibly infinite) of non compact smooth curves (without end-points). For instance, when $w(z_1,z_2)=z_1z_2$, $\{w=0\}$ is the union of two orthogonal straight lines.

Lemma 7.2.9. *Let $u^\pm=(a,u_2^\pm)\in\mathbb{R}^2$ be two wells, i.e. $W(u^\pm)=0$. Assume that, for all $z\in\mathbb{R}^2$, $W(z)=w^2(z)$ and for all $z\in[u^-,u^+]$, $w(z)\geq 0$ where $w:\mathbb{R}^2\to\mathbb{R}$ is locallly Lipschitz continuous and satisfies*

$$\partial_{11}w\pm\partial_{22}w=0$$

in the distributional sense. Then there exists $\Phi\in\mathcal{C}^1(\mathbb{R}^2,\mathbb{R}^3)$ and $\alpha\in\mathrm{Lip}_{loc}(\mathbb{R}^2)$ such that (7.2.7) and (7.2.9) hold. Moreover Φ can be chosen such that its Jacobian matrix is given by the following formula: $\forall z\in\mathbb{R}^2$,

$$\nabla\Phi(z)=\begin{pmatrix}-\alpha(z) & w(z)\\ \mp w(z) & -\alpha(z)\end{pmatrix},\tag{7.2.15}$$

where the convention $\nabla\Phi=(\partial_j\Phi_i)_{ij}$ (components on the rows and derivatives on the columns) is used.

Proof. By the Poincaré lemma, we know that there exists $\Phi=(\Phi_1,\Phi_2)$ such that (7.2.15) holds in the distributional sense if and only if the following system of two equations is satisfied by α and w:

$$-\partial_2\alpha-\partial_1 w=\pm\partial_2 w-\partial_1\alpha=0.$$

Then, the preceding system reads as follows:

$$\nabla\alpha=(\pm\partial_2 w,-\partial_1 w).\tag{7.2.16}$$

Consequently, applying once again the Poincaré lemma, the existence of a triplet (Φ,α) such that (7.2.15) holds is equivalent to the equation $\partial_{11}w\pm\partial_{22}w=0$ as stated in the lemma. Let Φ and α satisfying (7.2.15). Since w is, by assumption, locally Lipschitz continuous, so is α. In particular Φ is $\mathcal{C}^1$ as claimed in the lemma. Now, since $\partial_2\Phi_1(z) = w(z) = \sqrt{W(z)}$ for all $z \in [u^-, u^+]$, (7.2.9) is clearly satisfied. It remains to check (7.2.7). For all $u \in \mathcal{C}^\infty \cap L^\infty(\Omega, \mathbb{R}^2)$,

$$\begin{aligned} \nabla \cdot \{\Phi(u)\} + \alpha(u)\nabla \cdot u &= w(u)(\partial_1 u_2 \mp \partial_2 u_1) \\ &= \frac{1}{2}\Big\{(\partial_1 u_2 \mp \partial_2 u_1)^2 + W(u) - [w(u) - (\partial_1 u_2 \mp \partial_2 u_1)]^2\Big\} \end{aligned} \tag{7.2.17}$$

and (7.2.7) follows. □

Remark 7.2.10. As we said above, in the case where $\mp = -$, (7.2.15) exactly means that the function Φ defined by $\Phi(z) = \phi_1(x, y) + i\phi_2(x, y)$ for $z = x + iy \in \mathbb{C}$ is holomorphic and $-\Phi'(z) = \alpha(z) + iw(z)$. Then α is the harmonic conjugate (defined up to an additive constant) of w.

It is interesting to see that (7.2.17) exactly gives the defect in the inequality (7.2.7) integrated on Ω. In fact, integrating (7.2.17) on Ω and using regularization as we did in the proof of Proposition 7.2.5, we get by Proposition 7.2.4 that for all $u \in H^1_{loc}(\Omega, \mathbb{R}^2)$ satisfying $\nabla \cdot u = 0$, $E(u) < \infty$ and the boundary condition (7.2.3),

$$E(u) = \int_{u_2^-}^{u_2^+} \sqrt{W(a,t)} + \frac{1}{2}\int_\Omega (w(u) - \nabla \times u)^2 \tag{7.2.18}$$

when w is harmonic and

$$E(u) = \int_{u_2^-}^{u_2^+} \sqrt{W(a,t)} + \frac{1}{2}\int_\Omega [w(u) - (\partial_1 u_2 + \partial_2 u_1)]^2 + (\partial_1 u_1)^2 + (\partial_2 u_2)^2 \tag{7.2.19}$$

when w satisfies the wave equation. Here, we remind that $\int_{u_2^-}^{u_2^+} \sqrt{W(a,t)}$ is the infimum of the one-dimensional energy (see (7.1.8)). In particular, all global minimizer of E under both the divergence constraint and the boundary condition satisfy the equation

$$w(u) = \partial_1 u_2 \mp \partial_2 u_1 \text{ a.e. in } \Omega. \tag{7.2.20}$$

When w is harmonic we get $\nabla \times u = w(u)$. This observation is the main tool to prove the one-dimensional symmetry of global minimizers in the harmonic case. When w is a solution of the wave equation (i.e. $\pm = -$ in (7.2.2) and so $\mp = +$ in (7.2.20)), in addition to this condition, we get that $\partial_1 u_1 = \partial_2 u_2 = 0$ from which we can easily deduce the one-dimensional symmetry:

Theorem 7.2.11. *Let $u^\pm = (a, u_2^\pm) \in \mathbb{R}^2$ be two wells, i.e. $W(u^\pm) = 0$. Assume that, for all $z \in \mathbb{R}^2$, $W(z) \geq w^2(z)$ and for all $z \in [u^-, u^+]$, $W(z) = w^2(z)$ and $w(z) \geq 0$ where $w : \mathbb{R}^2 \to \mathbb{R}$ is solution of the wave equation, that is*

$$\frac{\partial^2 w}{\partial u_1^2} - \frac{\partial^2 w}{\partial u_2^2} = 0.$$

Then Question 1 holds true: every global minimizer of E under the divergence-free constraint and the boundary condition (7.2.3), if it exists, is unique up to a translation in the x_1 variable.

Remark 7.2.12. A global one-dimensional minimizer could fail to exist in which case the theorem becomes empty. The question of the existence of a global minimizer was independently established in the first section under some additional assumptions on W (see Proposition 7.1.2).

Proof. Let u be a global minimizer of E under the divergence free constraint and the boundary condition (7.2.3). Then by (7.2.19) we deduce that

$$\partial_1 u_2 + \partial_2 u_1 = w(u) \quad \text{and} \quad \partial_1 u_1 = \partial_2 u_2 = 0.$$

In particular, u_1 only depends on x_2. Thanks to Lemma 7.2.3, we know that $u_1(R_n, x_2) = u_1(x_2)$ converges uniformly to a for a sequence $R_n \to \infty$. Thus u_1 is constant: $u_1 \equiv a$. Moreover $u_2(x) = u_2(x_1)$ satisfies the ODE

$$\partial_1 u_2(x_1) = w(a, u_2(x_1)),$$

which characterizes the one-dimensional transition layer, unique up to translation. □

A fundamental example is the Aviles-Giga potential $W(z) = (1 - |z|^2)^2$. Note that $w(z) := 1 - |z|^2$ is invariant by rotation so that it satisfies the wave equation in every system of coordinates corresponding to an orthonormal basis. In [39], W. Jin and R. V. Kohn proved that the one-dimensional transition layer is a global minimizer while Theorem 7.2.11 states that it is the unique global minimizer up to translation.

We now treat the situation where w is harmonic. We get the a similar result:

Theorem 7.2.13. *Let* $u^{\pm} = (a, u_2^{\pm}) \in \mathbb{R}^2$ *be two wells, i.e.* $W(u^{\pm}) = 0$. *Assume that, for all* $z \in \mathbb{R}^2$, $W(z) \geq w^2(z)$ *and for all* $z \in [u^-, u^+]$, $W(z) = w^2(z)$ *and* $w(z) \geq 0$ *where* $w : \mathbb{R}^2 \to \mathbb{R}$ *is harmonic, that is*

$$\Delta w = \frac{\partial^2 w}{\partial u_1^2} + \frac{\partial^2 w}{\partial u_2^2} = 0.$$

Then Question 1 holds true: every global minimizer of E *under the divergence-free constraint and the boundary condition* (7.2.3), *if it exists, is unique up to a translation in the* x_1 *variable.*

Since (7.2.18) does not yield the strong condition $\partial_1 u_1 = \partial_2 u_2$, the proof is more complicated in this situation.

Proof. Let u be a global minimizer of E under the divergence-free constraint and the boundary condition (7.2.3). Then by (7.2.18) we deduce that $w(u) = \nabla \times u$ so that u is a solution of the following first order quasilinear system of PDE's:

$$\begin{cases} \nabla \cdot u & = 0 \\ \nabla \times u & = \mathrm{w(u)} \end{cases} \quad \text{a.e. in } \Omega. \tag{7.2.21}$$

Because of (7.2.18), the first order equation (7.2.21) is a characterization of global minimizers for E under both divergence and boundary conditions and, for this reason, it is stronger than the second order Euler-Lagrange equation.

Let compute the derivative of the second equation of (7.2.21) in the distributional sense with respect to the second variable:

$$-\partial_{22}u_1 + \partial_{12}u_2 = \partial_1 w\ \partial_2 u_1 + \partial_2 w\ \partial_2 u_2 \quad .$$

Since $\nabla \cdot u = 0$, we have $\partial_{12}u_2 = -\partial_{11}u_1$ and $\partial_2 u_2 = -\partial_1 u_1$ so that the preceding equation rewrites:

$$-\partial_{22}u_1 - \partial_{11}u_1 = \partial_1 w\ \partial_2 u_1 - \partial_2 w\ \partial_1 u_1 \quad .$$

Consequently, u_1 solves the following elliptic semi linear equation

$$-\Delta u_1 + \nabla^{\perp} w \cdot \nabla u_1 = 0. \tag{7.2.22}$$

In particular, since w is smooth, we deduce that u_1 is smooth using a classical Boot-strap argument for elliptic PDE's. We are going to conclude, using the classical maximum principal and our boundary condition (7.2.3), that u_1 is constant. Since $\int_\Omega |\nabla u|^2 \leq E(u) < \infty$, Lemma 7.2.3 allows to find a sequence $R_n \to \infty$ such that

$$u_1(\pm R_n, x_2) \to a \text{ uniformly in } \mathbb{T} \text{ when } n \to \infty \quad .$$

Let $\varepsilon > 0$ and $n \geq \frac{1}{\varepsilon}$ big enough so that $|u_1(R_n, x_2) - a| \leq \varepsilon$ for all $x_2 \in \mathbb{T}$. Applying the maximum principal to the elliptic equation (7.2.22) on the domain $[-R_n, R_n] \times \mathbb{T}$, we get:

$$\forall x \in [-R_n, R_n] \times \mathbb{T},\ |u_1(x) - a| \leq \varepsilon \quad .$$

Since this can be done for all $\varepsilon > 0$, we conclude that $u_1 \equiv a$. Thus $\nabla \cdot u = \partial_2 u_2 = 0$ and $u_(x) = u_(x_1)$ a.e. Moreover, as $w(u) = \nabla \times u = \partial_1 u_2$, $u_2(x_1)$ satisfies the ODE

$$\partial_1 u_2(x_1) = w(a, u_2(x_1)),$$

which characterizes the one-dimensional transition layer, unique up to translation. □

An elementary exemple of squared harmonic potential is given by $W(z) = (z_1 z_2)^2$. In this case, the set $\{W = 0\}$ is the union of the two orthogonal lines $\{z_1 = 0\}$ and $\{z_2 = 0\}$. For two distinct wells u^+ and u^- which are not on the same line ($\{z_1 = 0\}$ or $\{z_2 = 0\}$), the vector $\nu := u^+ - u^-$ can be any vector in $\mathbb{R}^2$ such that $W(\nu) \neq 0$. Theorem 7.2.13 asserts that, with a periodicity condition with respect to the second variable in the basis $(\nu^\perp, \nu)$, that is $x \cdot \nu$, we can deduce the symmetry for the optimal transition layer between u^- and u^+. Note that for two wells $u^- = (0, u_2^-)$ and $u^+ = (0, u_2^+)$ lying on the same line, in this case, $\{z_1 = 0\}$, then the global minimization problem has no solution. Indeed, for a one-dimensional vector field $u(x_1, x_2) = (0, \varphi(x_1))$ such that $\varphi(\pm\infty) = u_2^\pm$, the energy is given by

$$E(u) = \frac{1}{2} \int_{\mathbb{R}} |\varphi'(t)|^2 \, \mathrm{d}t \quad .$$

The infimum of this energy is then 0. Of course this infimum is not achieved if $u^- \neq u^+$ since, due to the Dirichlet energy, the only zero-energy configurations are given by constant vector fields.

It is interesting to remark that $W(z) = (z_1 z_2)^2$ is also the square of a solution of the wave equation since $\partial_{11}(z_1 z_2) = \partial_{22}(z_1 z_2) = 0$. However, as we noticed in Remark 7.2.2 page 124, it is preferable to consider $z_1 z_2$ as an harmonic function since the Laplace operator is invariant by rotation. For example, its rotation of angle $\frac{\Pi}{4}$, $z_1^2 - z_2^2$ is also harmonic but is not solution of the wave equation.

7.3 One-dimensional symmetry in higher dimension

We would want to generalize the method of the previous part in higher dimension. Namely, we look at divergence-free vector fields $u : \Omega \to \mathbb{R}^d$ where $d \geq 1$ and $\Omega := \mathbb{R} \times \mathbb{T}^{d-1}$. The energy of u is defined as

$$E(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 + W(u)\,\mathrm{d}x \quad , \tag{7.3.1}$$

where $W : \mathbb{R}^d \to \mathbb{R}^+$ is a nonnegative potential such that the set $\{W = 0\}$ is non empty. Moreover, we consider the following boundary condition on u:

$$\lim_{x_1 \to \pm\infty} u(x_1, x') = u^{\pm} \quad \text{for a.e. } x' \quad . \tag{7.3.2}$$

Note that this condition is stronger than that was considered in dimension two (see (7.2.3)). For technical reasons, this condition is required in the proof of our main symmetry result in higher dimension (see Theorem 7.3.3). We are interested in the one-dimensional symmetry of global minimizers of the energy (7.3.1) under the divergence-free and boundary conditions: namely, we want to prove that, under some assumptions on W, every global minimizer of (7.3.1) under the divergence-free constraint and (7.3.2) only depends on the first variable and is unique up to a translation. We will start by generalizing the entropy method in higher dimension in two different situations which are analogous to the case where the potential W was the square of some harmonic function or a solution of the wave equation. We will see that the extension of the harmonic case do not provide interesting examples although the extension of the case of a solution of the wave equation provides non trivial solutions. We will prove the one-dimensional symmetry in this situation (see Theorem 7.3.3) and then give an example of potential W for which the Theorem apply.

As in the preceding section we will need some growth condition on W. Namely, we will impose

$$\exists C > 0,\, \forall z \in \mathbb{R}^d,\, W(z) \leq C[1 + |z|^{2^*}], \tag{7.3.3}$$

where $2^* = \frac{2n}{n-2}$ is the critical Sobolev exponent, i.e.

$$\forall R > 0,\, \exists C > 0,\, \forall u \in H^1(\Omega_R),\, \|u\|_{L^{2^*}} \leq C\|u\|_{H^1(\Omega_R)},$$

where $\Omega_R = [-R, R] \times \mathbb{T}^{d-1}$. Since this is the only property we used to prove Lemma 7.2.6 and Proposition 7.2.5, this proposition can be generalized in higher dimension up to use our new growth condition (7.3.3). In particular Lemma 7.2.6 is true in every dimension and will be useful in the proof of our symmetry result. All what we need to apply the entropy method can be summarized in the following lemma:

Lemma 7.3.1. *Assume that W satisfies (7.3.3). Let us take $\Phi \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d)$ such that $|\nabla\Phi|^2$ satisfies the same growth condition:*

$$\exists C > 0,\, \forall z \in \mathbb{R}^d,\, |\nabla\Phi(z)|^2 \leq C[1 + |z|^{2^*}]. \tag{7.3.4}$$

Let $u \in H^1_{loc}(\Omega, \mathbb{R}^d)$ *satisfying our boundary condition* (7.3.2) *and assume that* $E(u) < \infty$. *Then* $\nabla \cdot \{\Phi(u)\} \in L^1(\Omega)$ *and*

$$\int_\Omega \nabla \cdot \{\Phi(u)\}\, \mathrm{d}x = \Phi_1(u^+) - \Phi_1(u^-).$$

The proof of this lemma is an easy adaptation of that of Lemma 7.2.6 and Proposition 7.2.5.

Entropy method in higher dimension Let denote by S_d^+ the space of symmetric matrices whose diagonal is a multiple of the identity matrix and S_d^- the space of all matrices obtained as the sum of an antisymmetric matrix and a multiple of the identity matrix:

$$S_d^\pm = \{U = (U_{i,j})_{i,j} \in \mathbb{R}^{d\times d} \ : \ U_{i,j} = \pm U_{j,i} \text{ if } i \neq j \text{ and } U_{1,1} = U_{2,2} = \cdots = U_{d,d}\}.$$

Then, denote by $\Pi^\pm$ the projection on the space $S_d^\pm$ for the usual Euclidean scalar product on $\mathbb{R}^{d\times d}$:

$$(U;V) = \mathrm{Tr}(UV^T) = \sum_{i,j} U_{i,j} V_{i,j} \quad ,$$

where U^T is the transpose matrix of U. It is easy to compute $\Pi^\pm U$ for a matrix $U \in \mathbb{R}^{n\times n}$:

$$(\Pi^\pm U)_{i,j} = \begin{cases} \frac{1}{n}\mathrm{Tr}(U) & \text{if } i = j, \\ \frac{U_{i,j} \pm U_{j,i}}{2} & \text{otherwise.} \end{cases}$$

Note that, thanks to Proposition 7.2.4, we have

$$E(u) = \frac{1}{2}\int_\Omega 2|\Pi^- \nabla u|^2 + W(u) = \frac{1}{2}\int_\Omega |\nabla \times u|^2 + W(u)$$

and

$$E(u) = \frac{1}{2}\int_\Omega 2|\Pi^+ \nabla u|^2 + W(u) + \int_\Omega \sum_i |u_{i,i}|^2$$

for all $u \in H^1_{loc}(\Omega)$ such that $\nabla \cdot u = \mathrm{Tr}(\nabla u) = 0$, $E(u) < \infty$ and u satisfies the boundary condition (7.3.2). In this framework, the entropy method consists in finding some locally Lipschitz maps $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ such that the entropy production $\int_\Omega \nabla \cdot \{\Phi(u)\}$ is controlled by the energy $E(u)$ and this estimate is sharp for the one-dimensional competitor (if it exists). The last condition ("saturation condition") means that, if u^{1D} is a minimizer of (7.1.8) (i.e. a one dimensional global minimizer under divergence and boundary conditions), then $\int_\Omega \nabla \cdot \{\Phi(u_{1D})\} = E(u_{1D}) = c_W(u^-, u^+)$. In general (eventually without existence of the one-dimensional transition layer), we will impose on Φ the following condition:

$$\Phi_1(u^+) - \Phi_1(u^-) = c_W(u^-, u^+), \tag{7.3.5}$$

where $u^\pm \in \mathbb{R}^d$ are two fixed wells: $W(u^\pm) = 0$. Let $\gamma : t \in [-1,1] \to \gamma(t) = (a, \psi(t)) \in \mathbb{R}^d$, be a geodesic from u^- to u^+ for the problem (7.1.7) that is

$$c_W(u^-, u^+) = \int_{-1}^1 \sqrt{W(\gamma(t))}|\gamma'(t)|\, \mathrm{d}t \ ; \ \gamma(\pm 1) = u^\pm \quad .$$

Then, if $\Phi \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R}^2)$, (7.3.5) rewrites as

$$\int_{-1}^{1} \nabla\Phi_1(\gamma(t)) \cdot \gamma'(t)\,\mathrm{d}t = \int_{-1}^{1} \sqrt{W(\gamma(t))}|\gamma'(t)|\,\mathrm{d}t \quad . \tag{7.3.6}$$

We have different strategies for the estimation of $\nabla \cdot \{\Phi(u)\}$ according to the choice we make for the first term of the energy density: $|\nabla u|^2$, $\frac{1}{2}|\Pi^- \nabla u|^2$ or $\frac{1}{2}|\Pi^+ \nabla u|^2$.

Estimate involving $|\nabla u|^2$: Strong punctual condition. The more natural choice is to impose the following density estimate for all $u \in L^\infty \cap \mathcal{C}^\infty(\Omega, \mathbb{R}^d)$ with $\nabla \cdot u = 0$:

$$\nabla \cdot \{\Phi(u)\} = (\nabla\Phi(u); \nabla u^T) \leq \frac{1}{2}\left(|\nabla u|^2 + W(u)\right). \tag{7.3.7}$$

As before, applying this inequality for a divergence-free vector field $u \in L^\infty \cap \mathcal{C}^\infty$ such that $u(x_0) = z$, $\nabla u(x_0) = p$ for some fixed point $x_0 \in \Omega$ and for all $z \in \mathbb{R}^d$ and $p \in \mathbb{R}^{d\times d}$ such that $\mathrm{Tr}(p) = 0$, we get

$$(\nabla\Phi(z); p^T) \leq \frac{1}{2}(|p|^2 + W(z)) \text{ whenever } \nabla\Phi(z) \text{ exists,}$$

which implies the following punctual condition on $\nabla\Phi$: for all $z \in \mathbb{R}^2$ such that $\nabla\Phi(z)$ exists and $W(z) \neq 0$,

$$\left|\nabla\Phi(z) - \frac{\mathrm{Tr}(\nabla\Phi(z))}{d} I_d\right| \leq \sqrt{W(z)}. \tag{7.3.8}$$

Indeed, let P_0 denote the projection on zero trace matrices in $\mathbb{R}^{d\times d}$: for all $U \in \mathbb{R}^{d\times d}$, $P_0 U = U - \frac{\mathrm{Tr}(U)}{d} I_d$. Then, for all $U \in \mathbb{R}^{d\times d}$, since $P_0 U =: p$ is a zero trace matrix, one has

$$(\nabla\Phi(z); P_0 U) = (P_0 \nabla\Phi(z); P_0 U) \leq \frac{1}{2}(|P_0 U|^2 + W(z)).$$

Now, if $P_0 \nabla\Phi(z) \neq 0$ (otherwise (7.3.8) is evident), we apply this inequality to $U = \lambda\nabla\Phi(z)$ with $\lambda \geq 0$ such that $\lambda^2 |P_0 \nabla\Phi(z)|^2 = W(z)$. We obtain

$$\lambda |P_0 \nabla\Phi(z)|^2 = |P_0 \nabla\Phi(z)| \sqrt{W(z)} \leq W(z),$$

which implies (7.3.8) since $W(z) \neq 0$.

However (7.3.7), coupled with (7.3.5) is often too strong in the applications. In particular, since $\gamma_1'(t) = 0$, (7.3.8) implies that $\nabla\Phi_1(z) \cdot \gamma'(t) \leq \sqrt{W(z)}|\gamma'(t)|$ for all $t \in [-1, 1]$. As a consequence, inequality (7.3.6) is saturated, i.e.

$$\nabla\Phi_1(\gamma(t)) = \sqrt{W(\gamma(t))}\,\frac{\gamma'(t)}{|\gamma'(t)|}, \quad \forall t \in [-1, 1].$$

Note that (7.3.8) is then saturated as well,

$$\nabla\Phi_2(\gamma(t)) = \nabla\Phi_3(\gamma(t)) = \cdots = \nabla\Phi_d(\gamma(t)) = 0, \quad \forall t \in [-1, 1],$$

and Φ is determined up to a constant on the geodesic γ.

Estimation with the curl operator: First weak condition. We want to extend the case of the preceding section where the potential was the square of some harmonic function in $2D$. It corresponds to impose the following density estimate for $u \in H^1_{loc}(\Omega)$ with $\nabla \cdot u = 0$:

$$\nabla \cdot \{\Phi(u)\} = (\nabla\Phi(u); \nabla u^T) \leq \frac{1}{2}\left(2|\Pi^- \nabla u|^2 + W(u)\right). \tag{7.3.9}$$

It is then natural to impose that $\nabla\Phi(z) = \Pi^- \nabla\Phi(z)$, i.e. $\nabla\Phi(z) \in S_d^-$ for all $z \in \mathbb{R}^d$. Indeed, with this property, we get

$$\begin{aligned}\nabla \cdot \{\Phi(u)\} &= (\nabla\Phi(u); \nabla u^T) = (\Pi^-\nabla\Phi(u); \nabla u^T) \\ &= (\nabla\Phi(u); \Pi^-\nabla u^T) = -(\nabla\Phi(u); (\Pi^-\nabla u)^T).\end{aligned}$$

Then, it is not difficult to see that (7.3.9) is equivalent to the following punctual condition: for all $z \in \mathbb{R}^d$,

$$\frac{1}{2}\sum_{i\neq j} |\partial_i \Phi_j(z)|^2 = \sum_{i<j} |\partial_i \Phi_j(z)|^2 \leq W(z). \tag{7.3.10}$$

Unfortunately, we are going to see that the condition $\nabla\Phi(z) \in S_d^-$ for all $z \in \mathbb{R}^d$ is too strong:

Proposition 7.3.2. *Let $d \geq 3$ be the dimension and $\Phi = (\Phi^1, \dots, \Phi^d) : \mathbb{R}^d \to \mathbb{R}^d$ be a locally Lipschitz map satisfying*

$$\forall z \in \mathbb{R}^d,\ \nabla\Phi(z) \in S_d^-. \tag{7.3.11}$$

Then there exists $c = (c_1, \dots, c_d) \in \mathbb{R}^d$ such that for all $i \in \{1, \dots, d\}$ and $z \in \mathbb{R}^d$,

$$\Phi^i(z) = \Phi^i(0) + \nabla\Phi^i(0)z + c_i\frac{|z_i|^2}{2} + \sum_{k\neq i}\left\{c_k z_k z_i - c_i\frac{|z_k|^2}{2}\right\}.$$

In dimension 2, (7.3.11) exactly matches with the ansatz we used in the previous part, that is (7.2.15) which is equivalent to say that Φ is holomorphic. However in higher dimension, the Proposition above says that we don't have "non trivial" solution anymore.

Proof. Up to regularize Φ by convolution with a smooth kernel, one can assume that Φ is smooth. Moreover, up to replace Φ by $\Phi - \Psi$ for $\Psi(z) = \Phi(0) + \nabla\Phi(0)z$, one can assume that $\Phi(0) = 0$ and $\nabla\Phi(0) = 0$. For the sake of simplicity, we are going to denote $f_i = \partial_i f$ for the derivative of some scalar or vector function f defined on $\mathbb{R}^d$. In particular, writing $\Phi = (\Phi^1, \dots, \Phi^n)$, we will note

$$\Phi^i_j = \partial_j \Phi^i \quad \text{for all } i, j \in \{1, \dots, d\}.$$

Now, (7.3.11) rewrites

$$\begin{cases} \Phi^1_1 = \cdots = \Phi^d_d := \alpha & , \\ \Phi^i_j = -\Phi^j_i & \text{for all } i \neq j. \end{cases}$$

In particular, if $i, j, k \in \{1, \dots, d\}$ are three distinct indices, then, using the Schwarz theorem, one gets

$$\Phi^i_{jk} = -\Phi^k_{ij} = \Phi^j_{ki} = -\Phi^i_{jk} \quad \text{and so} \quad \Phi^i_{jk} = 0.$$

In particular, Φ^i_j only depends on z_i and z_j. For the purpose on notation, let us note

$$\Phi^i_j(z) = \Phi^i_j(z_i, z_j) \quad \text{for } i \neq j.$$

Then, for $i \neq j$, one has

$$\Phi^i_{jj} = -\Phi^j_{ij} = -\alpha_i \quad \text{and} \quad \Phi^i_{ji} = \alpha_j \quad .$$

In particular, α_i only depends on z_i and z_j for all $j \neq i$. Since $d \geq 3$, for all $k \neq i$, α_i does not depend on z_k (it only depends on z_i and z_j for some j such that $j \neq k$ and $j \neq k$). Thus, α_i only depends on z_i:

$$\alpha_i(z) = \alpha_i(z_i).$$

Now, for $i \neq j$, one has

$$\Phi^i_{ijj} = (\Phi^i_{jj})_i = -\alpha_{ii} = (\Phi^i_{ij})_j = \alpha_{jj} \quad .$$

In particular, $-\alpha_{ii} = \alpha_{jj}$ for all $i \neq j$ which implies that $\alpha_{ii}(z) = 0$ for all $z \in \mathbb{R}^d$. Indeed, let us pick k such that $k \neq i$ and $k \neq j$. Then $-\alpha_{ii} = \alpha_{kk} = -\alpha_{jj} = \alpha_{ii}$ and so $\alpha_{ii} = 0$. Consequently, for all $i \in \{1, \dots, d\}$, α_i is constant:

$$\alpha_i \equiv c_i \in \mathbb{R}.$$

Since $\Phi^i_j(z) = \Phi^i_j(z_i, z_j)$ with $(\Phi^i_j)_i = \alpha_j = c_j$, $(\Phi^i_j)_j = -\alpha_i = -c_i$ and $(\Phi^i_i)_i = \alpha_i = c_i$ for all i, j such that $i \neq j$ and since $\nabla\Phi(0) = 0$, one has

$$\begin{cases} \Phi^i_j(z) = c_j z_i - c_i z_j & \text{for } i \neq j, \\ \Phi^1_1(z) = \cdots = \phi^d_d(z) = \alpha(z) = \sum_i c_i z_i \end{cases}$$

and the proposition follows. □

From Proposition 7.3.2 and (7.3.10), we learn that the only pertinent potentials for which we can characterize the minimal profile or deduce a rigidity property with this method are of the form $W(z) = \frac{1}{2}\sum_{i,j} |c_j - z_i - c_i z_j|^2$ for some $c = (c_1, \dots, c_d) \in \mathbb{R}^d$. In particular the set $\{W = 0\}$ is a vector subspace of $\mathbb{R}^d$ and the energy for the transition between two admissible wells is always 0 which is never achieved. As a result, the global minimization problem is of no interest since it has no solution.

Estimation with the symmetrized gradient: Second weak condition. We now extend the situation where the potential W is the square of some solution of the two dimensional wave equation. Namely, we will ask for the following estimate for all $u \in H^1_{loc}(\Omega)$ such that $\nabla \cdot u = 0$:

$$\nabla \cdot \{\Phi(u)\} \leq \frac{1}{2}\left(2|\Pi^+\nabla u|^2 + W(u)\right). \tag{7.3.12}$$

In order to get such an estimate we need to impose that $\nabla\Phi(z) \in S_d^+$ for almost every $z \in \mathbb{R}^d$. As above, if this property is satisfied, we will have

$$\nabla \cdot \{\Phi(u)\} = (\Pi^+\nabla\Phi(u); \nabla u^T) = (\nabla\Phi(u); (\Pi^+\nabla u)^T) = (\nabla\Phi(u); \Pi^+\nabla u) \qquad (7.3.13)$$

and, as above (7.3.12) is equivalent to the punctual condition (7.3.10). Since the diagonal of $\Pi^+\nabla u$ vanishes, only the norm of $\nabla\Phi$ outside its diagonal is needed in order to estimate $\nabla \cdot \{\Phi(u)\}$. More precisely, let Π_0 be the projection onto matrices whose diagonal vanishes: for $M \in \mathbb{R}^{d\times d}$ and $i, j \in \{1, \dots, d\}$,

$$(\Pi_0 M)_{ij} = \begin{cases} M_{ij} & \text{if } i \neq j, \\ 0 & \text{otherwise.} \end{cases}$$

Then one has

$$\begin{aligned}\nabla \cdot \{\Phi(u)\} &= (\Pi_0\nabla\Phi(u); \Pi^+\nabla u) \\ &\leq \frac{1}{2}\left(|\Pi_0\nabla\Phi(u)|^2 + |\Pi^+\nabla u|^2\right) = \sum_{i<j} |\partial_i\Phi_j(z)|^2 + \frac{1}{2}|\Pi^+\nabla u|^2.\end{aligned}$$

In general, we can prove the following theorem:

Theorem 7.3.3. *Assume that there exists* $\Phi \in W^{2,\infty}_{loc}(\mathbb{R}^d, \mathbb{R}^d)$ *such that the growth condition* (7.3.4) *is satisfied and* $\nabla\Phi(z) \in S_d^+$. *Let us consider the potential*

$$W(z) = \sum_{i<j} |\partial_i\Phi_j(z)|^2.$$

Let $u^\pm = (a, \overline{u}^\pm) \in \mathbb{R}^d$ *be two wells, that is* $W(u^\pm) = 0$. *Assume that there exists a one-dimensional solution* $v \in H^1_{loc}(\Omega; \mathbb{R}^d)$ *of the Euler-Lagrange equation*

$$2\,\Pi^+\nabla v = \Pi_0\nabla\Phi(v), \qquad (7.3.14)$$

such that $v(x) = v(x_1)$, $v_1 \equiv a$ *and* $v(\pm\infty) = u^\pm$.

Then Question 1 holds true: every global minimizer of E under the divergence-free constraint and the boundary condition (7.2.3) *is unique and coincide with v up to a translation in the* x_1 *variable.*

Note that, in this theorem, we do not use any existence theorem for global minimizers of E. In fact, the existence of an entropy is also useful to prove the existence and characterize the one-dimensional minimizer. The equation (7.3.14) rewrites as

$$\forall i, j \in \{2, \dots, n\},\ \partial_1 v_i(x_1) = \partial_1\Phi_i(v(x_1)) \quad \text{and} \quad \partial_j\Phi_i(v(x_1)) = 0 \quad \text{a.e.} \qquad (7.3.15)$$

Let denote by φ the $d-1$ last components of v, i.e. $\varphi(x_1) = (v_2(x_1), \dots, v_d(x_1))$. Then φ satisfies the EDO

$$\varphi'(t) = \partial_1[\Phi_2, \dots, \Phi_n](a, \varphi(t)) \quad \text{and} \quad \varphi(\pm\infty) = \overline{u}^\pm \quad .$$

Since $\partial_1\Phi$ is locally Lipschitz continuous, this EDO has a unique solution up to a translation.

Proof. Let u be a global minimizer of E under the divergence-free condition and (7.2.3). Then, (7.3.13) yields

$$\begin{aligned}\nabla\cdot\{\Phi(u)\} &= \left(\frac{1}{\sqrt{2}}\nabla\Phi(u)\ ;\ \sqrt{2}\,\Pi^+\nabla u\right) = \left(\frac{1}{\sqrt{2}}\Pi_0\nabla\Phi(u)\ ;\ \sqrt{2}\,\Pi^+\nabla u\right)\\ &= \frac{1}{2}\left(\frac{1}{2}|\Pi_0\nabla\Phi(u)|^2 + 2|\Pi^+\nabla u|^2 - \left(\frac{1}{\sqrt{2}}\Pi_0\nabla\Phi(u) - \sqrt{2}\,\Pi^+\nabla u\right)^2\right)\\ &= \frac{1}{2}\left(2|\Pi^+\nabla u|^2 + W(u)\right) - \frac{1}{4}\left(\Pi_0\nabla\Phi(u) - 2\,\Pi^+\nabla u\right)^2.\end{aligned} \tag{7.3.16}$$

Now, let us integrate this identity using Lemma 7.3.1:

$$\Phi_1(u^+) - \Phi_1(u^-) = \frac{1}{2}\int_\Omega 2|\Pi^+\nabla u|^2 + W(u) - \frac{1}{4}\int_\Omega\left(\Pi_0\nabla\Phi(u) - 2\,\Pi^+\nabla u\right)^2.$$

Thanks to Proposition 7.2.4, this inequality can be rewritten as

$$E(u) = \Phi_1(u^+) - \Phi_1(u^-) + \int_\Omega\left\{2\sum_i|\partial_i u_i|^2 + \frac{1}{4}\left(\Pi_0\nabla\Phi(u) - 2\,\Pi^+\nabla u\right)^2\right\}.$$

Note that, since the first term of the RHS of the preceding equation only depends on the boundary values $u^\pm$, minimizing E is equivalent to minimizing the second term of the RHS of this equation, that is

$$F(u) := \int_\Omega 2\sum_i|\partial_i u_i|^2 + \frac{1}{4}\left(\Pi_0\nabla\Phi(u) - 2\,\Pi^+\nabla u\right)^2.$$

By assumption, v achieve the minimum value of F, that is $F(v) = 0$. Since u is another global minimizer of F, it satisfies $F(u) = 0$, that is

$$\Pi_0\nabla\Phi(u) = 2\,\Pi^+\nabla u \quad\text{and}\quad \partial_i u_i = 0 \quad\text{for all}\quad i = 1,\dots,d \quad .$$

In particular, u_1 does not depend on x_1. Since $u_1(x) = u_1(x_2,\dots,x_d)$ tends to $a := u_1^+ = u_1^-$ as $x_1 \to \pm\infty$, we deduce that u_1 is constant: $u_1 \equiv a$.

Now, for all $y := (x_2,\dots,x_n) \in \mathbb{R}^{d-1}$, let $\varphi_y : \mathbb{R} \to \mathbb{R}^{d-1}$ be defined by $\varphi_y(x_1) = u(x_1; y)$. Then the energy of u rewrites as

$$\begin{aligned}E(u) &= \frac{1}{2}\int_{\mathbb{T}^{d-1}}\left\{\int_{\mathbb{R}}|\partial_{x_1}u(x_1,y)|^2 + W(u(x_1,y))\,\mathrm{d}x_1\right\}\mathrm{d}y + \frac{1}{2}\int_\Omega|\nabla_y u(x)|^2\,\mathrm{d}x\\ &= \frac{1}{2}\int_{\mathbb{T}^{d-1}}E^{1D}(\varphi_y)\,\mathrm{d}y + \frac{1}{2}\int_\Omega|\nabla_y u|^2,\end{aligned}$$

where ∇_y stands for the gradient with respect to the $d-1$ last variables $x_2,\dots,x_d$ and E^{1D} is defined in (7.1.5). Now, for a.e. $y \in \mathbb{T}^{d-1}$, we know that $\varphi_y(x_1) \to u^\pm$ when $x_1 \to \pm\infty$. Then $E^{1D}(\varphi_y) \geq c_W(u^-,u^+)$ and, since u is a global minimizer, we have $E^{1D}(\varphi_y) = c_W(u^-,u^+)$ for a.e. $y \in \mathbb{T}^{d-1}$. Thus

$$E(u) = c_W(u^-,u^+) + \int_\Omega|\nabla_y u|^2.$$

Once again, the minimality of u yields $\nabla_y u = 0$ a.e. in Ω. Consequently u only depends on x_1. □

Now, let us look for examples of applications $\Phi \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d)$ such that $\nabla\Phi(z) \in S_d^+$ a.e. Note that, if this condition holds then, by the Poincaré Lemma, there exists $\Psi \in \mathcal{C}^2(\mathbb{R}^d)$ such that for all $z \in \mathbb{R}^d$,

$$\Phi(z) = \nabla\Psi(z),$$

where $\nabla^2\Psi$ is the Hessian matrix of Ψ. Moreover, by definition of S_d^+, we must have

$$\partial_1\Phi_1 = \cdots = \partial_d\Phi_d$$

or, equivalently

$$\partial_{11}\Psi = \cdots = \partial_{dd}\Psi.$$

In other words, we look for scalar functions $\Psi \in \mathcal{C}^2(\mathbb{R}^d)$ satisfying the following degenerate system of PDE's

$$\partial_{ii}\Psi = \partial_{jj}\Psi \quad \text{for all } i, j \in \{1, \dots, d\}.$$

By analogy with the wave equation in $\mathbb{R}^2$, all solutions of this equations can be represented as follows

$$\Psi(z) = \sum_{\varepsilon \in \{\pm 1\}^d} f_\varepsilon(\varepsilon \cdot z),$$

where $(f_\varepsilon)_\varepsilon$ is a family of scalar functions defined on $\mathbb{R}$. Let us give an example in dimension 3 given by a fourth order polynomial

$$\Psi(z_1, z_2, z_3) = z_1 z_2 \left(\frac{z_1^2 + z_2^2}{3} + z_3^2 - 1 \right).$$

In this case one has

$$\partial_{ii}\Psi(z) = 2z_1z_2 \quad \text{for} \quad i = 1, 2, 3,$$

and, defining the entropy $\Phi := -\nabla\Psi$, one has

$$\nabla\Phi(z) = -\nabla^2\Psi(z) = \begin{pmatrix} -2z_1z_2 & 1 - |z|^2 & -2z_2z_3 \\ 1 - |z|^2 & -2z_1z_2 & -2z_1z_3 \\ -2z_2z_3 & -2z_1z_3 & -2z_1z_2 \end{pmatrix}. \tag{7.3.17}$$

Since each component of $\nabla\Phi$ is a polynomial of degree 2 and $2^* = 6$ in dimension 3, Φ clearly satisfies the growth condition (7.3.4). Then, we can apply Theorem 7.3.3 for the corresponding potential

$$W(z) = (|z|^2 - 1)^2 + 4z_3^2(z_1^2 + z_2^2).$$

Then, the set $\{W = 0\}$ reads

$$\{z \in \mathbb{R}^3 \ : \ W(z) = 0\} = \{z \in S^2 \ : \ z_3 = 0 \quad \text{or} \quad z_1 = z_2 = 0\} = S^1 \cup \{\pm e_3\},$$

where S^2 is the unit sphere in $\mathbb{R}^3$, $S^1 = S^2 \cap \{z_3 = 0\}$ and (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$. Let $u^\pm = (a, \overline{u}^\pm)$ be two wells in S^1 such that $(u^+ - u^-) \cdot e_1 = 0$:

$$u^\pm = (a, \pm b, 0),$$

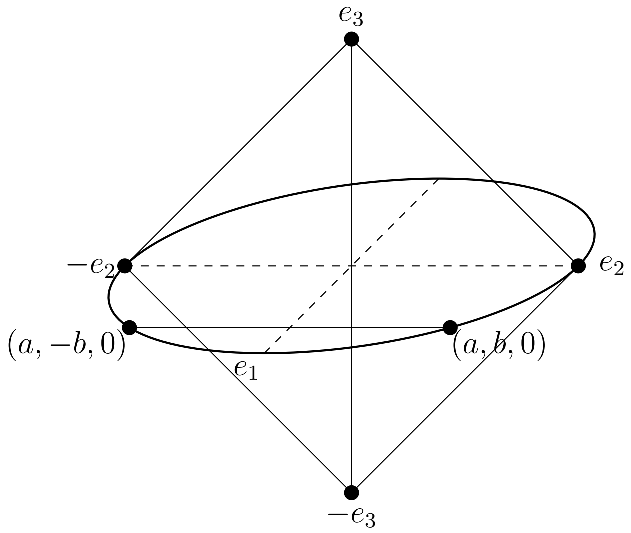

Figure 7.3 – The set $\{W = 0\}$

where $b > 0$ and $(a, b) \in S^1$, i.e. $a^2 + b^2 = 1$ (see figure 7.3).

Note that W is invariant by rotation in the plane $\{z_3 = 0\}$ so that the fact that we consider two wells such that $u^+ - u^-$ is orthogonal to e_1 is not restrictive.

Let us study the existence of a one-dimensional minimizer, i.e. $v(x) = v(x_1) = (a, v_2(x_1), v_3(x_1))$ satisfying the Euler-Lagrange equation (7.3.15). Let us take $\varphi(x_1) = (\varphi_1(x_1), \varphi_2(x_1)) := (v_2(x_2), v_3(x_1))$, then this equation reads

$$\varphi'(x_1) = \big(1 - a^2 - |\varphi(x_1)|^2 \ ; \ -2\varphi_1(x_1)\varphi_2(x_1)\big) \quad \text{and} \quad a\,\varphi_2(x_1) = 0. \tag{7.3.18}$$

Consequently, if $a \neq 0$, $\varphi_2 \equiv 0$. In fact, even in the case where $a = 0$, one can show that the only solutions connecting u^- and u^+ must satisfy $\varphi_2 \equiv 0$ and so

$$\varphi_1' = b^2 - \varphi_1^2 = (b - \varphi_1)(b + \varphi_1).$$

This ODE has two stationary solutions $\varphi = \pm b$ and a unique solution φ, up to a translation, such that $\varphi(\pm\infty) = \pm b$. In other words, the geodesic v follows a straight line from u^- to u^+ (see figure 7.4).

It remains to understand the transition between two wells in the set $\{\pm e_2, \pm e_3\}$, i.e. $a = 0$. By symmetry, it is enough to study transitions from $u^- = e_3$ to $u^+ = e_2$ and from $u^- = e_3$ to $u^+ = -e_3$. (7.3.18) is of the form

$$\varphi' = (1 - |\varphi|^2, -2\varphi_1\varphi_2) \quad \varphi(\pm\infty) = \overline{u}^{\pm}.$$

For the transition between e_3 and e_2, it is convenient to make the change of variable $\psi = (\psi_1, \psi_2) := (\varphi_1 + \varphi_2, \varphi_1 - \varphi_2)$ so that the preceding ODE becomes

$$(\psi_1', \psi_2') = (1 - \psi_1^2 \ ; \ 1 - \psi_2^2) \quad ; \quad \psi(\pm\infty) = (u_2^{\pm} + u_3^{\pm} \ ; \ u_2^{\pm} - u_3^{\pm}),$$

which is decoupled. Once again, the only solution connecting e_3 and e_2 is a straight line (see figure 7.4). More precisely, in this case, one has $\psi(-\infty) = (1, -1)$, $\psi(+\infty) = (1, 1)$ and so

$$\psi_1 = \varphi_1 + \varphi_2 \equiv 1 \quad \text{and} \quad \psi_2(x_1) = \varphi_1(x_1) - \varphi_2(x_1) = \tanh(x_1).$$

For the transition between e_3 and $-e_3$, we remark that the line $\{z_1 = z_3 = 0\}$ is the reunion of five solutions of the ODE (7.3.18): two stationary solutions $\varphi \equiv (\pm 1, 0)$, one supported on $(-\infty, -1) \times \{0\}$, one on $(-1, 1) \times \{0\}$ and the other on $(1, +\infty) \times \{0\}$. In particular, by the Cauchy-Lipshitz Theorem, any solution can go across the line $\{z_1 = z_3 = 0\}$ and there do not exist a solution connecting $-e_3$ and e_3.

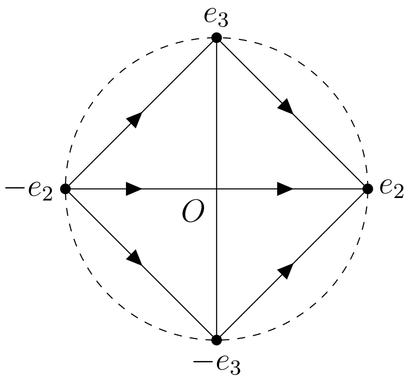

Figure 7.4 – Geodesics in the plane $\{z_1 = 0\}$

Chapter 8

Lower bound for Aviles-Giga type functionals

Our aim is to illustrate how looking for an entropy is helpful to achieve the $\Gamma-\liminf$ part for the Γ-convergence of some free energies. This method is well known since the work of P. Aviles and Y. Giga (see [7]) and has been generalized to several situations (see [36] and [38] for instance). In these works, the existence of an entropy is used to prove the $\Gamma - \liminf$ in some Ginzburg-Landau type models. All these models involve a potential W defined on $\mathbb{R}^2$ or on the 2-dimensional sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ in micromagnetics applications and vanishing on a circle $\mathbb{S}^1 \subset \mathbb{R}^2$ or $\mathbb{S}^2 \cap \{z_3 = 0\}$. For instance, in the Aviles-Giga model, the potential reads $W(z) = (1-|z|^2)^2$ which is the classical Ginzburg-Landau potential. Our aim is to generalize the procedure in proving the $\Gamma - \liminf$ in the context of a general potential W. Our main assumption will be the existence of an entropy or a family of entropies which is a very difficult problem in general. As an application we will establish a Γ-convergence type theorem for potentials which are the square of an harmonic function on $\mathbb{R}^2$ (see Theorems 8.2.1 and 8.3.2). This will be a direct application of the previous chapter, chapter 7.

Let $d \geq 2$ be the dimension and $\Omega \subset \mathbb{R}^d$ be a bounded open subset. We consider the functional

$$E_\varepsilon(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega \varepsilon|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\,\mathrm{d}x & \text{if } u \in H^1(\Omega,\mathbb{R}^d) \text{ and } \nabla\cdot u = 0, \\ +\infty & \text{otherwise,} \end{cases} \tag{8.0.1}$$

where $W : \mathbb{R}^d \to \mathbb{R}^+$ is a nonnegative continuous potential. For the sake of simplicity, we will also use the following notation for the energy density:

$$d_\varepsilon(u) = \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{2\varepsilon}W(u).$$

Remark 8.0.4. Here, $|\cdot|$ stands for the matrix norm induced by the Euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^d$:

$$\forall A \in \mathbb{R}^{d\times d},\ |A| = \sup_{\|x\|_2\leq 1} \|Ax\|_2.$$

This is not the standard choice in this kind of models. Indeed, it is more usual to take the Euclidean norm $|\cdot|_2$ on $\mathbb{R}^{d\times d}$ defined by $|A|_2^2 := \sum_{ij} A_{ij}^2$ for all $A = (A_{ij})_{1\leq i,j\leq d} \in \mathbb{R}^{d\times d}$.

However, note that one has $|A| \leq |A|_2$ for all matrices A and $|A| = |A|_2$ whenever A is of the form $A = u \otimes v$ for some $u, v \in \mathbb{R}^d$. Indeed,

$$|u \otimes v| = \sup_{\|x\|_2 \leq 1} \|(u \otimes v)x\|_2 = \sup_{\|x\|_2 \leq 1} |v \cdot x| \|u\|_2 = \|u\|_2 \|v\|_2 = |u \otimes v|_2.$$

Our aim, in chapter 8, is to get some estimates from below of the energy E_ε which are sharp for the one-dimensional transition layers (see Theorem 8.2.1). Since $|\cdot| \leq |\cdot|_2$, all the estimates we will get are still valid if one replaces $|\cdot|$ by $|\cdot|_2$ in the definition of E_ε. Moreover, for a one-dimensional transition layer of the form $u(x) = \gamma(\nu \cdot x)$, with $\gamma : \mathbb{R} \to \mathbb{R}^d$ and $\nu \in \mathbb{S}^{d-1}$, one has $\nabla u(x) = \dot{\gamma}(\nu \cdot x) \otimes \nu$ and so $|\nabla u| = |\nabla u|_2$: for a one-dimensional transition layer u, the definition of $E_\varepsilon(u)$ does not change when replacing $|\cdot|$ by $|\cdot|_2$. In particular, all estimates from below of E_ε which are sharp for the one-dimensional transition layers are still sharp for $|\cdot|_2$ instead of $|\cdot|$. For technical reasons, we prefer to use the induces norm $|\cdot|$.

As in chapter 7, we assume the following polynomial growth condition on W:

$$\exists C > 0,\ \forall z \in \mathbb{R}^d,\ W(z) \leq C \cdot \begin{cases} 1 + |z|^p & \text{for some } p \geq 1 \text{ if } d = 2, \\ 1 + |z|^{2^*} & \text{where } 2^* = \frac{2d}{d-2} \text{ if } d \geq 3. \end{cases} \tag{8.0.2}$$

Note that, when $d \geq 3$, $2^* > 2$ is the critical exponent leading to the Sobolev embedding $H^1(\mathbb{R}^d) \hookrightarrow L^{2^*}(\mathbb{R}^d)$. We remind that this condition is sufficient to insure density in energy of smooth functions (see Lemma 7.2.6). More precisely, in our situation, one has

Lemma 8.0.5. *Assume that $u \in H^1_{loc}(\Omega)$ has finite energy, i.e. $\nabla \cdot u = 0$ and $E_\varepsilon(u) < \infty$ for some fixed $\varepsilon > 0$. Under assumption* (8.0.2)*, there exists a divergence-free sequence of smooth and bounded functions $(u_k)_{k \geq 0} \subset \mathcal{C}^\infty(\Omega) \cap L^\infty(\Omega)$ converging to u in $H^1_{loc}(\Omega)$ such that*

$$E_\varepsilon(u_k) \underset{k \to \infty}{\longrightarrow} E_\varepsilon(u).$$

It is well known that the energy $E_\varepsilon(u)$ of some vector field u tends to concentrate on hypersurfaces, i.e. $(d-1)$-rectifiable subsets of $\mathbb{R}^d$ when $\varepsilon \to 0$. In other words free discontinuity problems are expected in the limit. In order to estimate E_ε from below by some free discontinuity energy, we use the entropy method as described in the following section.

8.1 Notion of “entropy” and associated cost function

Throughout chapter 8, we will consider entropies $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ which are locally Lipschitz continuous. Since Φ is non globally Lipschitz continuous, the entropy production, $\nabla \cdot [\Phi(u)]$ might not make sense for unbounded u. However, thanks to Lemma 8.0.5, this is not a problem since one can approximate finite energy structures u by a sequence of smooth and bounded vector fields.

8.1.1 Definitions

Definition 8.1.1. Let $q \geq 1$ some fixed exponent and $W : \mathbb{R}^d \to \mathbb{R}^+$ a continuous potential. $\Phi \in \mathrm{Lip}_{loc}(\mathbb{R}^d)$ is called a $q-$entropy (or entropy if no confusion is possible) if the two following conditions are satisfied:

1. There exists $C > 0$ such that
$$|\Phi(z)| \leq C(1+|z|^q) \quad \text{for all } z \in \mathbb{R}^d. \tag{8.1.1}$$
2. Let $\Omega \subset \mathbb{R}^d$ be an open subset. For all divergence-free sequence $(u_\varepsilon)_{\varepsilon>0} \subset \mathcal{C}^\infty \cap L^\infty(\Omega, \mathbb{R}^d)$ such that $E_\varepsilon(u_\varepsilon) + \|u_\varepsilon\|_{L^q(\Omega)}$ is bounded and for all $\chi \in \mathcal{D}(\Omega)$, one has
$$\langle \nabla \cdot [\Phi(u_\varepsilon)] \,; \chi \rangle \leq \langle d_\varepsilon(u_\varepsilon) \,; \chi \rangle + \underset{\varepsilon \to 0}{\mathrm{o}}(1), \tag{8.1.2}$$
The last term in the right hand side of (8.1.2) is a sequence $(R_\varepsilon(\chi))_{\varepsilon>0}$ which may depend on χ such that $R_\varepsilon(\chi) \underset{\varepsilon \to 0}{\longrightarrow} 0$ for all $\chi \in \mathcal{D}(\Omega)$.

Here, $\langle f \,, g \rangle$ denotes the standard duality product between functions, $\langle f \ ; g \rangle = \int_\Omega f(x) g(x) \, \mathrm{d}x$. The set of all entropies will be denoted by $\mathcal{E}^q_W(\mathbb{R}^d)$ (or $\mathcal{E}(\mathbb{R}^d)$ if no confusion is possible) on the exponent q and the potential W which is considered.

Remark 8.1.2. Note that, in (8.1.1), the constant $C > 0$ could depend on Φ while the exponent q is uniform on $\mathcal{E}^q_W(\mathbb{R}^d)$.

Remark 8.1.3. The set $\mathcal{E}^q_W(\mathbb{R}^d)$ is non-decreasing with respect to q: if $1 \leq q_1 \leq q_2$ then $\mathcal{E}^{q_1}_W(\mathbb{R}^d) \subset \mathcal{E}^{q_2}_W(\mathbb{R}^d)$.

We now want to introduce the notion of free energy associated to an entropy set. In section 5.2, we introduced a large class of free discontinuity problems with divergence constraint. Given some cost function $f : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1} \to \mathbb{R}^+$ such that
$$f(u^+, u^-, \nu) = f(u^-, u^+, -\nu) = f(u^+, u^-, -\nu), \tag{8.1.3}$$
we remind the definition of E_f: for $u \in \mathrm{BV}(\Omega, \mathbb{R}^d)$,
$$E_f(u) = \begin{cases} \displaystyle\int_{J(u)} f\left(u^+(x), u^-(x), \nu_u(x)\right) \mathrm{d}\mathcal{H}^{d-1}(x) & \text{if } \nabla \cdot u = 0, \\ +\infty & \text{otherwise,} \end{cases} \tag{8.1.4}$$
where ν is the orientation of the jump set $J(u)$ of u and $u^\pm$ its traces. In chapter 6, we saw that no non-trivial sufficient conditions on f for the l.s.c. of E_f is known, even in dimension 2. Fortunately, the only free energies of this type that we are going to consider here will be de facto l.s.c. as supremum of l.s.c. functionals.

We use a generalization in every dimension of the formalism introduced in dimension 2 by R. Ignat and B. Merlet in [37]. For all entropies subset $\Phi \subset \mathcal{E}(\mathbb{R}^d)$, we consider the cost function induced by Φ: for all $u^\pm \in \mathbb{R}^d$ and $\nu \in \mathbb{S}^{d-1}$,
$$c_\Phi(u^-, u^+, \nu) := \begin{cases} \overline{c_\Phi}(u^-, u^+, \nu) & \text{if } W(u^\pm) = 0 \text{ and } (u^+ - u^-) \cdot \nu = 0, \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\overline{c_\Phi}(u^-,u^+,\nu) := \sup\{[\Phi(u^+)-\Phi(u^-)]\cdot\nu \ :\ \Phi\in\boldsymbol{\Phi}\} \quad \text{for all } u^\pm\in\mathbb{R}^d,\ \nu\in\mathbb{S}^{d-1}.$$

Note that the cost function $c_{\boldsymbol{\Phi}}$, as well as the notion of entropy in Definition 8.1.1 (which involves the energy density d_ε), depends on W. As to the constraint $(u^+-u^-)\cdot\nu=0$, it comes from the divergence constraint $\nabla\cdot u$ (see Lemma 5.2.1). We will say that $\boldsymbol{\Phi}\subset\mathcal{E}(\mathbb{R}^d)$ is symmetric if one has

$$\forall\Phi\in\boldsymbol{\Phi},\ -\Phi\in\boldsymbol{\Phi}.$$

In the sequel, we are going to restrict to entropy sets $\boldsymbol{\Phi}$ which are symmetric. Note that if $\boldsymbol{\Phi}$ is symmetric, then the associated cost function $c_{\boldsymbol{\Phi}}$ is nonnegative and satisfies (8.1.3) (see Proposition 8.1.9). Moreover, as supremum of l.s.c. functions, $c_{\boldsymbol{\Phi}}$ is l.s.c. (see Proposition 8.1.6). Our aim is to get some estimates from below of the energies E_ε involving the energies E_f for $f=c_{\boldsymbol{\Phi}}$ with $\boldsymbol{\Phi}\subset\mathcal{E}(\mathbb{R}^d)$. For convenience of the reader, we will use the notation $E_{\boldsymbol{\Phi}}:=E_{c_{\boldsymbol{\Phi}}}$. An interest of these free energies associated to some entropy subset $\boldsymbol{\Phi}$ is that they are automatically lower semicontinuous in the following sense

Proposition 8.1.4. *Let $\boldsymbol{\Phi}\subset\mathcal{E}(\mathbb{R}^d)$ be a symmetric entropy subset. Then, for all $u\in\mathrm{BV}(\Omega,\mathbb{R}^d)$ and for every sequence $(u_n)_{n\geq1}$ converging to u in $L^1(\Omega,\mathbb{R}^d)$, one has*

$$E_{\boldsymbol{\Phi}}(u)\leq\liminf_{n\to\infty}E_{\boldsymbol{\Phi}}(u_n),$$

where $E_{\boldsymbol{\Phi}}$ has been defined in (8.1.4).

Remark 8.1.5. This property is almost equivalent to say that $E_{\boldsymbol{\Phi}}$ is l.s.c. Actually, one can extend $E_{\boldsymbol{\Phi}}$ to $L^1(\Omega,\mathbb{R}^d)$ by $+\infty$ outside $\mathrm{BV}(\Omega,R^d)$ and then define the relaxation of $E_{\boldsymbol{\Phi}}$ in L^1 (for the L^1 convergence). Denoting this relaxation by $\overline{E}_{\boldsymbol{\Phi}}$ we get that $\overline{E}_{\boldsymbol{\Phi}}(u)=E_{\boldsymbol{\Phi}}(u)$ for every $u\in\mathrm{BV}(\Omega,\mathbb{R}^d)$.

Proof. The proof is an easy adaptation in every dimension of that of Theorem 3 in [37]. The main tool is the following representation formula: for all $u\in\mathrm{BV}(\Omega,\mathbb{R}^d)$,

$$E_{\boldsymbol{\Phi}}(u)=\sup\left\{\sum_{i=1}^n\langle\Phi_i(u)\,;\nabla\varphi_i\rangle\ :\ n\geq1,\ \Phi_i\in\boldsymbol{\Phi},\ \varphi_i\in\mathcal{C}_c^\infty(\Omega,\mathbb{R}^+),\ \sum_{i=1}^n\varphi_i\leq1\right\}.$$

It is clear that, if this formula is satisfied then $E_{\boldsymbol{\Phi}}$ is l.s.c. as supremum of l.s.c. functionals. □

8.1.2 Regularity and symmetry of cost functions associated with an entropy subset

The first trivial property which is satisfied by all cost functions $c_{\boldsymbol{\Phi}}$ is the l.s.c.

Proposition 8.1.6. *Let $\boldsymbol{\Phi}\subset\mathcal{E}(\mathbb{R}^d)$. Then $c_{\boldsymbol{\Phi}}$ is l.s.c. on $\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}$.*

Proof. Let $X := \{(u^+,u^-,\nu) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1} \ : \ W(u^+) = W(u^-) = 0$ and $(u^+ - u^-) \cdot \nu = 0\}$. Since W is continuous, X is closed and the indicator function I_X in the sense of convex analysis (0 in X and $+\infty$ elsewhere) is l.s.c. Moreover, $\overline{c_\Phi}$ is l.s.c. as supremum of l.s.c. functions. In particular, $c_\Phi = I_X + \overline{c_\Phi}$ is l.s.c. □

In case where the functions $\Phi \in \Phi$ are locally uniformly Lipschitz, one can prove that c_Φ is also Lipschitz continuous on its domain, as claimed in Proposition 8.1.7.

Proposition 8.1.7. *Assume that the functions $\phi \in \Phi$ are locally uniformly Lipschitz continuous and bounded: for all compact subset $K \in \mathbb{R}^d$, there exists $C > 0$ such that* $\sup\{\phi(z) \ : \ \phi \in \Phi, \ z \in K\} \leq C$ *and $L > 0$ such that*

$$\sup\{|\phi(z_1) - \phi(z_2)| \ : \ z_1, \, z_2 \in K \ \textit{and} \ \phi \in \Phi\} \leq L|z_1 - z_2|.$$

Then c_Φ is locally Lipschitz continuous on its domain, $\mathcal{D}(c_\Phi) := \{T = (u^+,u^-,\nu) \ : \ (u^+ - u^-) \cdot \nu = 0$ and $W(u^\pm) = 0\}$.

Proof. Let $T_1 = (u_1^+, u_1^-, \nu_1)$, $T_2 = (u_2^+, u_2^-, \nu_2) \in K \times K \times \mathbb{S}^{d-1}$ be two triplets such that T_1, $T_2 \in \mathcal{D}(c_\Phi)$. Up to exchange T_1 and T_2, it is enough to prove that $c_\Phi(T_1) \leq c_\Phi(T_2) + L'|T_1 - T_2|$ for some constant L'. By assumption, one has

$$c_\Phi(T_1) = \sup_{\phi \in \Phi} c_{\{\phi\}}(T_1) \leq c_\Phi(T_2) + \sup_{\phi \in \Phi} \left[c_{\{\phi\}}(T_1) - c_{\{\phi\}}(T_2)\right],$$

which implies the claim since $c_{\{\phi\}}$ is uniformly Lipschitz continuous on $K^2 \times \mathbb{S}^{d-1}$ for $\phi \in \Phi$. Indeed, since $c_{\{\phi\}}(T_i) = [\phi(u_i^+) - \phi(u_i^-)] \cdot \nu_i$ for $i = 1$ or 2, there exists $L' > 0$ such that

$$\begin{aligned} |c_{\{\phi\}}(T_1) - c_{\{\phi\}}(T_2)| &= |[\phi(u_1^+) - \phi(u_1^-)] \cdot \nu_1 - [\phi(u_2^+) - \phi(u_2^-)] \cdot \nu_2| \\ &\leq |\nu_1| \left[|\phi(u_1^+) - \phi(u_2^+)| + |\phi(u_1^-) - \phi(u_2^-)|\right] + 2C|\nu_1 - \nu_2| \\ &\leq L'|T_1 - T_2|. \end{aligned}$$

□

We know explore the invariance properties of c_Φ in function of the symmetry of the set Φ. Since Φ is a subset of $\mathcal{E}(\mathbb{R}^d)$, we first have to explore the invariances properties of $\mathcal{E}(\mathbb{R}^d)$, depending on W. Let $\mathcal{O}(d) := \{\sigma \in \mathcal{L}(\mathbb{R}^d) \ : \ \|\sigma(z)\|_2 = \|z\|_2$ for all $z \in \mathbb{R}^d\}$ denote the orthogonal group. For $\sigma \in \mathrm{GL}(d)$ and $F : \Omega \subset \mathbb{R}^d \to \mathbb{R}^d$ we denote by $\sigma \cdot F : \sigma(\Omega) := \{\sigma(x) \ : \ x \in \Omega\} \to \mathbb{R}^d$ the action by conjugation of σ over F:

$$\sigma \cdot F(x) = \sigma F(\sigma^{-1} x) \quad \text{for } x \in \sigma(\Omega).$$

We will identify matrices $M \in \mathbb{R}^{d \times d}$ to the endomorphism $(z \to Mz) \in \mathcal{L}(\mathbb{R}^d)$ so that we keep the same notation $\sigma \cdot M = \sigma M \sigma^{-1}$ for the action by conjugation on matrices.

Proposition 8.1.8. 1. *$\mathcal{E}(\mathbb{R}^d)$ is symmetric: $-\mathcal{E}(\mathbb{R}^d) = \mathcal{E}(\mathbb{R}^d)$.*

2. *Let $\sigma \in \mathcal{O}(d)$. Assume that W is invariant by σ, i.e. $W \circ \sigma = W$. Then, $\mathcal{E}(\mathbb{R}^d)$ is invariant under the action of σ:*

$$\forall \Phi \in \mathcal{E}(\mathbb{R}^d), \, \sigma \cdot \Phi \in \mathcal{E}(\mathbb{R}^d).$$

Proof. We first prove the second statement. It is clear that (8.1.1) is stable under the action of $\mathcal{O}(d)$. It remains to prove (8.1.2). Let us fix $\Phi \in \mathcal{E}(\mathbb{R}^d) = \mathcal{E}^q_W(\mathbb{R}^d)$, $\sigma \in \mathcal{O}(d)$ and $v \in \mathcal{C}^\infty \cap L^\infty(\mathbb{R}^d)$. We define $u := \sigma^{-1} \cdot v$. Then,

$$\begin{aligned}\forall x \in \mathbb{R}^d,\ \nabla \cdot \{(\sigma \cdot \Phi) \circ v\}(x) &= \mathrm{Tr}\{\sigma \nabla \Phi(\sigma^{-1} v(x)) \sigma^{-1} \nabla v(x)\} \\ &= \mathrm{Tr}\{\sigma \nabla \Phi[u \circ \sigma^{-1}(x)] \sigma^{-1} \nabla[\sigma \cdot u](x)\} \\ &= \mathrm{Tr}\{\sigma \nabla \Phi[u \circ \sigma^{-1}(x)] \sigma^{-1} \sigma \nabla u(\sigma^{-1} x) \sigma^{-1}\} \\ &= \mathrm{Tr}\{\nabla \Phi[u \circ \sigma^{-1}(x)] \nabla u(\sigma^{-1} x)\} \\ &= \nabla \cdot [\Phi \circ u](\sigma^{-1}(x)).\end{aligned}$$

Let Ω be a bounded open subset of $\mathbb{R}^d$. Let $(u_\varepsilon)_{\varepsilon>0} \subset \mathcal{C}^\infty \cap L^\infty(\Omega, \mathbb{R}^d)$ be such that $\|u_\varepsilon\|_{L^q} + E_\varepsilon(u_\varepsilon)$ is bounded. Let $v_\varepsilon := \sigma \cdot u_\varepsilon$ be defined on the bounded open subset $\sigma(\Omega) \subset \mathbb{R}^d$. Since $\Phi \in \mathcal{E}^q_W(\mathbb{R}^d)$ and $\sigma \in \mathcal{O}(d)$, for all $\chi \in \mathcal{D}(\mathbb{R}^d)$, one has

$$\langle \nabla \cdot \{(\sigma \cdot \Phi) \circ v_\varepsilon\}\,; \chi \rangle = \langle \nabla \cdot [\Phi \circ u_\varepsilon]\,; \chi \circ \sigma \rangle \leq \langle d_\varepsilon(u_\varepsilon) \circ \sigma^{-1}\,; \chi \rangle + \underset{\varepsilon \to 0}{\mathrm{o}}(1).$$

Finally, since $|\sigma A| = |A|$ for $A \in \mathbb{R}^{d \times d}$ and $W(\sigma \cdot) = W(\cdot)$, one has

$$\begin{aligned} d_\varepsilon(v_\varepsilon)(x) &= \frac{1}{2}\left\{\varepsilon |\nabla v_\varepsilon(x)|^2 + \frac{1}{\varepsilon} W(v_\varepsilon(x))\right\} \\ &= \frac{1}{2}\left\{\varepsilon |\sigma \nabla u_\varepsilon(\sigma^{-1}(x)) \sigma^{-1}|^2 + \frac{1}{\varepsilon} W(\sigma u_\varepsilon(\sigma^{-1} x))\right\} \\ &= d_\varepsilon(u_\varepsilon) \circ \sigma^{-1}(x),\end{aligned}$$

which implies that $\sigma \cdot \Phi \in \mathcal{E}(\mathbb{R}^d)$. The proof of the first statement, $-\mathcal{E}(\mathbb{R}^d) = \mathcal{E}(\mathbb{R}^d)$ is similar. Indeed, defining $v_\varepsilon(x) = u_\varepsilon(-x)$ (instead if $v_\varepsilon = \sigma \cdot u_\varepsilon$) and $\xi(x) = \chi(-x)$). Then, for $\Phi \in \mathcal{E}(\mathbb{R}^d)$, one has

$$\langle \nabla \cdot [-\Phi \circ v_\varepsilon]\,; \chi \rangle = \langle \nabla \cdot [\Phi \circ u_\varepsilon]\,; \xi \rangle \leq \langle d_\varepsilon(u_\varepsilon)\,; \xi \rangle + \underset{\varepsilon \to 0}{\mathrm{o}}(1) = \langle d_\varepsilon(v_\varepsilon)\,; \chi \rangle + \underset{\varepsilon \to 0}{\mathrm{o}}(1). \qquad \square$$

Proposition 8.1.9. *Assume that $\Phi \subset \mathcal{E}(\mathbb{R}^d)$ is symmetric.*

1. *c_Φ is nonnegative and $c_\Phi(u^+, u^-, \nu) = c_\Phi(u^+, u^-, -\nu) = c_\Phi(u^-, u^+, \nu)$ for all $u^\pm \in \mathbb{R}^d$, $\nu \in \mathbb{S}^{d-1}$.*
2. *If $d = 2$, then c_Φ depends on $u^\pm$ only through the constraint $\nu \cdot (u^+ - u^-)$: $c_\Phi(u^+, u^-, \nu) = c_\Phi(u^+, u^-)$ for all $u^\pm \in \mathbb{R}^d$, $\nu \in \mathbb{S}^{d-1}$ such that $(u^+ - u^-) \cdot \nu = 0$ and $W(u^\pm) = 0$.*
3. *Let $G \subset \mathcal{O}(d)$ be a subgroup of the orthogonal group such that W is invariant by G: $W \circ \sigma = W$ for all $\sigma \in G$. Assume also that Φ is invariant under the action of G by conjugation: $\sigma \cdot \Phi \in \Phi$ for all $\sigma \in G$ and $\Phi \in \Phi$. Then, for all $\sigma \in G$, one has*
$$c_\Phi(\sigma(u^+), \sigma(u^-), \sigma(\nu)) = c_\Phi(u^+, u^-, \nu).$$

Proof. The first statement is a direct consequence of the symmetry condition $\Phi = -\Phi$. The second statement follows from the fact that the condition $(u^+ - u^-) \cdot \nu$ determines $\pm\nu \in \mathbb{S}^{d-1}$. For the third statement, note that, for all isometry $\sigma \in G$, $T = (u^+, u^-, \nu) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}$ satisfies the constraint, $W(u^\pm) = 0$ and $(u^+ - u^-) \cdot \nu$, if and only if

$\sigma T := (\sigma(u^+), \sigma(u^-), \sigma(\nu))$ satisfies the same constraint. In case where $(u^+ - u^-) \cdot \nu$, the third statement is a consequence of the following trivial computation: for $\Phi \in \boldsymbol{\Phi}$,

$$[\sigma \cdot \Phi(\sigma(u^+)) - \sigma \cdot \Phi(\sigma(u^-))] \cdot \sigma(\nu) = [\Phi(u^+) - \Phi(u^-)] \cdot \nu \quad . \qquad \square$$

Exemple 8.1.10. The case of the Ginzburg-Landau potential $W(z) = (1 - |z|^2)^2$ was studied in dimension 2 by P. Aviles and Y. Giga in [7] and W. Jin and R. V. Kohn in [39]. They used the following polynomial entropy

$$\Phi_0(z) = 2 \left[z_2(1 - z_1^2) - \frac{z_2^3}{3} \; ; \; z_1(1 - z_2^2) - \frac{z_1^3}{3} \right] \quad \text{for all } z \in \mathbb{R}^2.$$

Since W is invariant by rotation, the set $\mathcal{E}(\mathbb{R}^2)$ is also invariant by conjugation with a rotation. Then one can define $\boldsymbol{\Phi} \subset \mathcal{E}(\mathbb{R}^2)$ the entropy set generated by Φ_0:

$$\boldsymbol{\Phi} := \langle \Phi_0 \rangle := \{\pm \sigma \cdot \Phi_0 \ : \ \sigma \in \mathcal{O}(2)\}.$$

In this case, the authors shows that the associated cost takes the following form

$$c_{\boldsymbol{\Phi}}(u^+, u^-, \nu) = \frac{|u^+ - u^-|^3}{3} \quad .$$

In the article [37] where the notion of cost function induced by an entropy set $\boldsymbol{\Phi}$ was introduced, the authors studied the question of the l.s.c. of line energies independently of the approximating energy sequence E_ε. They restricted to cost functions f defined on $\mathbb{S}^1$ and only depend on $|u^+ - u^-|$: $f(u^+, u^-, \nu) = g(|u^+ - u^-|)$ for $u^\pm, \nu \in \mathbb{S}^1$ such that $u^+ \cdot \nu = u^- \cdot \nu$ and $+\infty$ elsewhere. Note that this is the situation expected for potentials W which vanish on $\mathbb{S}^1$ and are positive elsewhere. In [37], the authors used the notion of entropy introduced by A. DeSimone, S. Müller, R. V. Kohn and F. Otto in [28]: $\Phi \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}^2)$ is said to be a entropy on $\mathbb{S}^1$ if for every $z = e^{i\theta} \in \mathbb{S}^1$,

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \Phi(z) \cdot z = 0,$$

where $\frac{\mathrm{d}}{\mathrm{d}\theta}$ stands for the angular derivative: $\frac{\mathrm{d}}{\mathrm{d}\theta}\Phi(z) = \frac{\mathrm{d}}{\mathrm{d}\theta}\left[\Phi(e^{i\theta})\right]$. In particular, it was shown that the cost function induced by a finite set of entropies, $\langle \Phi_1, \ldots, \Phi_k \rangle$ is at least cubic, $c_{\langle \Phi_1, \ldots, \Phi_k \rangle} = \mathrm{O}(t^3)$ (see Proposition 11 in [37]). However, by using an infinite set of entropies, the authors were able to show the l.s.c. for the quadratic cost $g(t) = t^2$.

In section 8.3, we give examples of entropy sets $\boldsymbol{\Phi}$ generated by a single entropy Φ_0 for potentials W such that the set $\{W = 0\}$ is not $\mathbb{S}^1$. Namely, we study potentials of the form $W = w^2$ for some harmonic function $w : \mathbb{R}^2 \to \mathbb{R}$.

8.1.3 Saturation condition

Our definition of an entropy, Definition 8.1.1, is based on an estimation of the entropy production by the energy E_ε. This will provide an estimate from above of $E_{\boldsymbol{\Phi}}$ by the energy: $E_{\boldsymbol{\Phi}} \leq \Gamma \liminf_{\varepsilon \to 0} E_\varepsilon$ (see Theorem 8.2.1). For this estimate to be sharp, we need an additional condition, called saturation condition (see Definition 8.1.11). This condition insures that the estimate $E_\Phi \leq \liminf_{\varepsilon\to 0} E_\varepsilon$ is sharp for the one-dimensional transition layer. We first remind the definition of the cost c_W induced by W, defined as the infimum of the energy $E_{\varphi=1}$ for $1D$ transition layers: for all $u^\pm \in \mathbb{R}^d$, $\nu \in \mathbb{S}^{d-1}$ such that $u^+\cdot\nu = u^-\cdot\nu =: a \in \mathbb{R}$,

$$c_W(u^+,u^-,\nu) := \inf_{\gamma\in H^1_{loc}(\mathbb{R},\mathbb{R}^d)} \left\{ \frac{1}{2}\int_{\mathbb{R}} |\dot\gamma(t)|^2 + W(\gamma(t))\,\mathrm{d}t \;:\; \gamma(\pm\infty) = u^\pm \text{ and } \gamma\cdot\nu \equiv a \right\}.$$

Here the constraint $\gamma\cdot\nu = a$ comes from the divergence constraint: the vector field u defined by $u(x) = \gamma(x\cdot\nu)$ is divergence-free if and only if $\gamma\cdot\nu = cte$ everywhere. It is clear that $c_W(u^+,u^-) = +\infty$ whenever $W(u^+) \neq 0$ or $W(u^-) \neq 0$. Indeed, if $\int_{\mathbb{R}} W(\gamma(t))\,\mathrm{d}t < +\infty$, then $W(\gamma(t))$ converges to $0 = W(u^\pm)$ as $|t| \to \infty$ since W is continuous. Moreover, the change of variables $t = \frac{s}{\varepsilon}$, $\gamma_\varepsilon(s) := \gamma(t)$ yields

$$c_W(u^+,u^-,\nu) := \inf_{\gamma_\varepsilon\in H^1_{loc}(\mathbb{R},\mathbb{R}^d)} \left\{ E^{1D}_\varepsilon(\gamma_\varepsilon) \;:\; \gamma_\varepsilon(\pm\infty) = u^\pm \text{ and } \gamma_\varepsilon\cdot\nu \equiv a \right\},$$

where $E^{1D}_\varepsilon(\gamma_\varepsilon) := \frac{1}{2}\int_{\mathbb{R}} \varepsilon|\dot{\gamma_\varepsilon}(s)|^2 + \frac{1}{\varepsilon}W(\gamma(s))\,\mathrm{d}s$. In dimension 2 the expression of c_W can be simplified in

$$c_W(u^+,u^-,\nu) = c_W(u^+,u^-) := |u^+ - u^-| \int_0^1 \sqrt{W((1-t)u^- + tu^+)}\,\mathrm{d}t \quad .$$

Definition 8.1.11. Let $\Phi \subset \mathcal{E}(\mathbb{R}^d)$. We say that Φ satisfies the saturation condition if one has

$$c_\Phi(u^+,u^-,\nu) = c_W(u^+,u^-,\nu) \quad \text{for all } u^\pm \in \mathbb{R}^d,\ \nu \in \mathbb{S}^{d-1} \quad .$$

Remark 8.1.12. From the definition of c_Φ (resp. c_W), we deduce that $c_\Phi(u^+,u^-,\nu) < +\infty$ (resp. $c_W(u^+,u^-,\nu) < +\infty$) if and only if $W(u^\pm) = 0$ and $(u^+ - u^-)\cdot\nu = 0$. For this reason, the condition $c_\Phi = c_W$, is restrictive only for admissible triplets (u^+,u^-,ν), i.e. $W(u^\pm) = 0$ and $(u^+ - u^-)\cdot\nu = 0$.

Remark 8.1.13. In dimension $d = 2$, c_Φ as well as c_W does not depend on ν. By contrast, in dimension $d \geq 3$, c_W may depend on ν. Indeed, $\nu \in (u^+ - u^-)^\perp$ appears in the constraint of the minimization problem which defines c_W. As illustrated in figure 8.1, only paths which are contained in the hyperplane $H_\nu := \{x \in \mathbb{R}^d \;:\; x\cdot\nu = a\}$ are admissible.

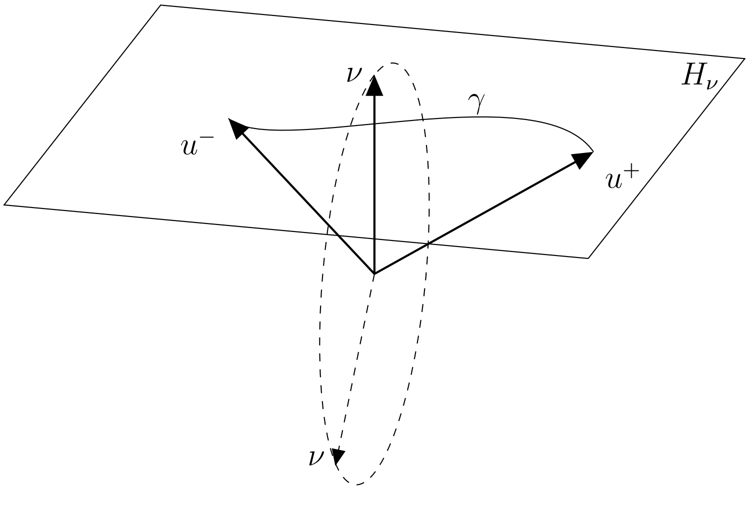

Figure 8.1 – Admissible paths from u^- to u^+ in the hypperplane plane $H_\nu := \{x\cdot\nu = a\}$

8.2 Main result: lower bound on energies $(E_\varepsilon)_{\varepsilon>0}$

Let $E_{1D} := E_{c_W}$ be the free energy associated to the cost function $f = c_W$. As explained above, the existence of an entropy set Φ satisfying the saturation condition allows to prove the optimality of the one-dimensional transition layer in the limit when $\varepsilon \to 0$:

Theorem 8.2.1. *Let $W = \mathbb{R}^d \to \mathbb{R}^+$ be a continuous potential satisfying* (8.0.2) *and $\Phi \subset \mathcal{E}^q_W(\mathbb{R}^d)$ be a symmetric entropy set where $q \geq 1$ is some fixed exponent. For all sequence $(u_\varepsilon)_{\varepsilon>0} \subset H^1_{loc}(\Omega, \mathbb{R}^d)$ strongly converging in L^q to $u \in \mathrm{BV}(\Omega, \mathbb{R}^d)$, one has*

$$E_\Phi(u) \leq \liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon). \tag{8.2.1}$$

In other words $E_\Phi \leq \Gamma - \liminf_{\varepsilon\to 0} E_\varepsilon$ on $\mathrm{BV}(\Omega, \mathbb{R}^d)$. In particular, if Φ satisfies the saturation condition, one has $E_\Phi = E_{1D}$ and so

$$E_{1D} \leq \Gamma - \liminf_{\varepsilon\to 0} E_\varepsilon \quad \textit{on } \mathrm{BV}(\Omega, \mathbb{R}^d). \tag{8.2.2}$$

Remark 8.2.2. The inequality (8.2.2) is sharp in the sense that it is an equality for the one-dimensional transition layers. More precisely, the $\Gamma - \limsup$ property is satisfied when u is the following trivial BV structure. Let $(u^\pm, \nu) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}$ such that $(u^+ - u^-) \cdot \nu = 0$ and $W(u^\pm) = 0$. Let σ_ν be a rotation sending $e_1 = (1, 0, \ldots, 0)$ on ν. Let us consider the domain $\Omega_\nu := \sigma_\nu\left((-1,1)^d\right)$ and define the $1D$ transition on Ω_ν by

$$u_{1D} = \begin{cases} u^+ & \text{if } x \cdot \nu \geq 0, \\ u^- & \text{if } x \cdot \nu < 0. \end{cases}$$

Then, there exists a recovery sequence $(u_\varepsilon)_\varepsilon$ for u_{1D}, i.e. $(u_\varepsilon)_\varepsilon$ converging to u_{1D} in L^q such that

$$E_\varepsilon(u_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} E_\Phi(u_{1D}).$$

Indeed, one can take $u_\varepsilon(x) = \gamma_\varepsilon(x \cdot \nu)$, where γ_ε is chosen in an optimal way, that is such that $E^{1D}_\varepsilon(\gamma_\varepsilon) = c_W(u^+, u^-, \nu)$. However, the general $\Gamma - \limsup$ inequality remains open in this general context, even for limiting BV configurations: we are not able to find a recovery sequence for all $u \in \mathrm{BV}(\Omega, \mathbb{R}^d)$. Nevertheless, we think that, as it holds for the Aviles-Giga model, the $\Gamma - \limsup$ is true for every $u \in \mathrm{BV}$. Note that, even for the classical Aviles-Giga functional, the $\Gamma - \limsup$ inequality remains open for limiting configurations which are not in BV.

Remark 8.2.3. In the second part of Theorem 8.2.1, it is not necessary to assume that Φ satisfies the saturation condition. It is enough to assume that $c_\Phi \geq c_W$. In fact as a consequence of (8.2.1) and remark 8.2.2, one can prove that

$$\forall \Phi \subset \mathcal{E}(\mathbb{R}^d),\ c_\Phi \leq c_W \quad .$$

Indeed, applying (8.2.1) to $u = u_{1D}$ and $(u_\varepsilon)_{\varepsilon>0}$, the recovery sequence of the preceding remark on the domain $\Omega = (-1,1)^d$, yields $2\,c_\Phi(u^+, u^-, \nu) \leq 2\,c_W(u^+, u^-, \nu)$.

Proof. The entropy method was used for the first time by P. Aviles and Y. Giga in [7] to derive such a $\Gamma-\liminf$ property in the case of the Ginzburg-Landau potential, $W(z)=(1-|z|^2)^2$. We make a slightly different proof which uses the entropy method and classical tools of geometric measure theory. The proof is divided into several steps. We first use cut-off functions in the target space $\mathbb{R}^d$ so as to consider only (pseudo-)entropies which are compactly supported. Then we use cut-off functions in the initial space Ω so as to localize our estimate. Finally we conclude by a classical argument in geometric measure theory using derivation of measures.

Let $(u_\varepsilon)_{\varepsilon>0}\subset H^1_{loc}(\Omega,\mathbb{R}^d)$ be a sequence strongly converging to $u\in \mathrm{BV}(\Omega,\mathbb{R}^d)$ in L^q. One can assume that $(u_\varepsilon)_{\varepsilon>0}$ is a bounded energy sequence:

$$\exists C>0,\, \forall \varepsilon>0,\ E_\varepsilon(u_\varepsilon)\le C.$$

Moreover, thanks to Lemma 8.0.5, which can be applied since W satisfies the growth condition (8.0.2), one can assume that each u_ε is smooth and bounded. Let denote by $\mu_\varepsilon\in\mathcal{M}(\Omega)$ the energy density of u_ε:

$$\mathrm{d}\mu_\varepsilon(x):=d_\varepsilon(u_\varepsilon)(x)\,\mathrm{d}\mathcal{L}^d(x)=\frac{1}{2}\left[\varepsilon|\nabla u_\varepsilon|^2+\frac{1}{\varepsilon}W(u_\varepsilon)\right]\mathrm{d}\mathcal{L}^d(x).$$

Since $E_\varepsilon(u_\varepsilon)\le C<\infty$, the sequence $(\mu_\varepsilon)_{\varepsilon>0}$ is bounded in the set of finite measures $\mathcal{M}(\Omega)$. Consequently, up to extraction, one can assume that $(\mu_\varepsilon)_{\varepsilon>0}$ weakly converges in $\mathcal{M}(\Omega)$:

$$\mu_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \mu\in\mathcal{M}(\Omega)\quad \text{weakly as measures.}$$

Finally, since $(u_\varepsilon)_\varepsilon$ converges in L^q, one can assume, up to extraction, that $(u_\varepsilon)_{\varepsilon>0}$ converges almost everywhere.

We first assume that Φ is countable. In the last step of the proof, we will see that this is not restrictive.

First step: Localization of entropies Given $K>0$, we define a cut-off function $\xi_K\in\mathcal{C}^\infty_c(\mathbb{R}^d,\mathbb{R}^+)$ satisfying

$$\mathbf{1}_{B(0,K)}\le \xi_K\le \mathbf{1}_{B(0,K+2)}\quad\text{and}\quad \|\nabla\xi_K\|_{L^\infty}\le 1.$$

For each $\Phi\in\Phi$, let us define Φ_K by

$$\forall z\in\mathbb{R}^d,\ \Phi_K(z)=\xi_K(z)\Phi(z).$$

Note that Φ_K is Lipschitz continuous. Moreover, for all $\Phi\in\Phi$ and $\chi\in\mathcal{C}^\infty_c(\Omega)$, one has

$$\begin{aligned}\langle\nabla\cdot[\Phi_K(u_\varepsilon)]\,;\chi\rangle &= \langle\Phi(u_\varepsilon)\,;\nabla\chi\rangle+\langle\Phi_K(u_\varepsilon)-\Phi(u_\varepsilon)\,;\nabla\chi\rangle\\ &\le \langle\nabla\cdot[\Phi(u_\varepsilon)]\,;\chi\rangle+\langle(\xi_K(u_\varepsilon)-1)\Phi(u_\varepsilon)\,;\nabla\chi\rangle\\ &\le \langle d_\varepsilon(u_\varepsilon)\,;\chi\rangle+R_\varepsilon(\chi)+I(K)\|\nabla\chi\|_{L^1}\quad ,\end{aligned}$$

where the sequence $(R_\varepsilon(\chi))_{\varepsilon>0}$ converges to 0 as $\varepsilon\to 0$ (see Definition 8.1.1) and $I(K)=\sup_{\varepsilon>0}\int_{|u_\varepsilon|>K}|\Phi(u_\varepsilon(x))|\,\mathrm{d}x$. It is easy to see that $\lim\limits_{K\to\infty} I(K)=0$, i.e. $(\Phi(u_\varepsilon))_\varepsilon$

is uniformly integrable. Indeed, thanks to (8.1.1), one has $|\Phi(u_\varepsilon)| \le C(1+|u_\varepsilon|^q)$ and, by assumption, the sequence $(u_\varepsilon)_\varepsilon$ converges in L^q. In brief, for all $\Phi \in \Phi$, $\chi \in \mathcal{C}_c^\infty(\Omega)$, the preceding equation yields

$$\langle \Phi_K(u_\varepsilon)\, ;\nabla\chi\rangle \le \langle \mu_\varepsilon\, ;\chi\rangle + R_\varepsilon(\chi) + I(K)\|\nabla\chi\|_{L^1(\Omega)},$$

where we remind that μ_ε is the absolute continuous measure such that $\mathrm{d}\mu_\varepsilon(x) = d_\varepsilon(u_\varepsilon)\,\mathrm{d}\mathcal{L}^d(x)$. It is clear that each term of the preceding inequality converges as $\varepsilon \to 0$. Indeed, since Φ_K is Lipschitz continuous and u_ε converges to u, $\Phi_K(u_\varepsilon)$ strongly converges to $\Phi_K(u)$ in $L^1(\Omega,\mathbb{R}^d)$ as well. Moreover, by assumption, μ_ε weakly converges to μ as measures. Finally, Definition 8.1.1 requires that $R_\varepsilon(\chi)$ converges to 0. Hence,

$$\langle \Phi_K(u)\, ;\nabla\chi\rangle \le \langle \mu\, ;\chi\rangle + I(K)\|\nabla\chi\|_{L^1} \quad . \tag{8.2.3}$$

It is easy to pass to the limit when $K \to \infty$. On the one hand, $(\Phi_K \circ u)_{K>0}$ converges to $\Phi \circ u$ as $K \to \infty$. Indeed, $\|\Phi_K(u) - \Phi(u)\|_{L^1} \le \int_{\{x\,:\,|u(x)|\ge K\}} |\Phi(u(x))|\,\mathrm{d}x$ which converges to 0 as $K \to \infty$ since $\Phi \circ u \in L^1(\Omega)$. On the other hand $I(K) \to 0$ as $K \to \infty$ and one gets

$$\langle \nabla\cdot[\Phi(u)]\, ;\chi\rangle \le \langle \mu\, ;\chi\rangle. \tag{8.2.4}$$

Second step: localization in space of the estimate Let us take $x_0 \in J(u)$ and $r > 0$ small enough so that $B(x_0,r) \subset \Omega$. Let $B_0 := B(0,1)$ be the unit ball and fix a cut-off function $\chi_0 \in \mathcal{C}_c^\infty(B_0)$ such that $\|\chi_0\|_{L^\infty} \le 1$. Now, we define $\chi_{x_0,r}$ on $B(x_0,r) = x_0 + rB_0$ by

$$\forall x \in \mathbb{R}^d,\ \chi_{x_0,r}(x) := \chi_0\left(\frac{x-x_0}{r}\right).$$

We want to apply the inequality (8.2.3) for the specific test function $\chi = \chi_{x_0,r}$. Then, the first term of (8.2.3) reads

$$\begin{aligned}\langle \Phi_K(u)\, ;\nabla\chi_{x_0,r}\rangle &= \int_{B(x_0,r)} \Phi_K(u(x))\cdot\frac{1}{r}\nabla\chi_0\left(\frac{x-x_0}{r}\right)\mathrm{d}x \\ &= r^{d-1}\int_{B_0} \Phi_K(u(x_0+rx))\cdot\nabla\chi_0(x)\,\mathrm{d}x.\end{aligned}$$

Let us define the blow-up sequence $u_{x_0,r} \in \mathrm{BV}(B_0)$ by $u_{x_0,r}(x) = u(x_0+rx)$ for all $x \in B_0$. We know that for $\mathcal{H}^{d-1}$-a.e. point $x_0 \in J(u)$, $u_{x_0,r} \xrightarrow[r\to 0]{} u_{x_0,0}$ in $L^1(B_0)$, where $u_{x_0,0}$ is defined by

$$\forall x \in B_0,\ u_{x_0,0}(x) = \begin{cases} u^+ & \text{if } x \in B_{0,\nu}^+ := \{x \in B_0 \ :\ x\cdot\nu \ge 0\}, \\ u^- & \text{if } x \in B_{0,\nu}^- := \{x \in B_0 \ :\ x\cdot\nu < 0\}. \end{cases}$$

where $\nu := \nu_u(x_0)$. Indeed this is just a reformulation of the definition of traces of a BV function, i.e. (5.2.2). Since Φ_K is Lipschitz continuous, one has also $\Phi_K \circ u_{x_0,r} \xrightarrow[r\to 0]{} \Phi_K \circ u_{x_0,0}$ in $L^1(B_0)$ for $\mathcal{H}^{d-1}$-a.e. point $x_0 \in J(u)$. In particular, for $\mathcal{H}^{d-1}$-a.e. $x \in J(u)$,

$$\int_{B_0} \Phi_K(u(x_0+rx))\cdot\nabla\chi_0(x)\,\mathrm{d}x \xrightarrow[r\to 0]{} \int_{B_0} \Phi_K(u_{x_0,0})\cdot\nabla\chi_0(x)\,\mathrm{d}x \quad .$$

Now, an integration by parts and the definition of $u_{x_0,0}$ yield

$$\begin{aligned}\int_{B_0} \Phi_K(u_{x_0,0}) \cdot \nabla\chi_0(x)\,\mathrm{d}x &= \int_{B^+_{0,\nu}} \Phi_K(u^+) \cdot \nabla\chi(x)\,\mathrm{d}x + \int_{B^-_{0,\nu}} \Phi_K(u^-) \cdot \nabla\chi(x)\,\mathrm{d}x \\ &= -\int_{H_{0,\nu}} \chi(x)\,\mathrm{d}\mathcal{H}^{d-1}(x)[\Phi_K(u^+) - \Phi_K(u^-)] \cdot \nu \quad ,\end{aligned}$$

where $H_{0,\nu} := \overline{B}^+_{0,\nu} \cap \overline{B}^-_{0,\nu}$. Note that, up to replace Φ by $-\Phi$ which is possible since Φ is symmetric, one can get rid of the minus sign in the preceding equation. It remains to estimate the last term in (8.2.3) for $\chi = \chi_{x_0,r}$, i.e. $I(K)\|\nabla\chi\|_{L^1}$. One has

$$\|\nabla\chi_{x_0,r}\|_{L^1} = \int_{B(x_0,r)} \frac{1}{r}|\nabla\chi_0|\left(\frac{x-x_0}{r}\right)\mathrm{d}x = r^{d-1}\|\nabla\chi_0\|_{L^1(B_0)}.$$

Now dividing (8.2.3) by r^{d-1} and applying the estimates above finally yield

$$\left\{\int_{H_{0,\nu}} \chi_0(x)\,\mathrm{d}\mathcal{H}^{d-1}(x)\right\}[\Phi_K(u^+) - \Phi_K(u^-)] \cdot \nu \le \liminf_{r\to 0} \frac{\langle \mu\,;\chi_{x_0,r}\rangle}{r^{d-1}} + I(K)\|\nabla\chi_0\|_{L^1(B_0)}.$$

Now, in the limit when $K \to \infty$, one gets

$$\left\{\int_{H_{0,\nu}} \chi_0(x)\,\mathrm{d}\mathcal{H}^{d-1}(x)\right\}[\Phi(u^+) - \Phi(u^-)] \cdot \nu \le \liminf_{r\to 0} \frac{\langle \mu\,;\chi_{x_0,r}\rangle}{r^{d-1}} \le \limsup_{r\to 0} \frac{\mu(B(x_0,r))}{r^{d-1}}$$

since $\|\chi_{x_0,r}\|_{L^\infty} \le 1$ and $\operatorname{supp}(\chi_{x_0,r}) \subset B(x_0,r)$. Then, we take the supremum over all cut-off functions χ_0 such that $\chi_0 \in \mathcal{C}^\infty_c(B_0)$ and $\|\chi_0\|_{L^\infty} \le 1$. From the monotone convergence Theorem, it is clear that the supremum value of the first term of the left hand side in the preceding equation is $\mathcal{H}^{d-1}(H_{0,\nu}) =: \omega_{d-1}$, measure of the unit ball of $\mathbb{R}^{d-1}$. Thus,

$$[\Phi(u^+) - \Phi(u^-)] \cdot \nu \le \liminf_{r\to 0} \frac{\mu(B(x_0,r))}{\omega_{d-1}r^{d-1}} \quad .$$

Now, since countable union of negligible sets for $\mathcal{H}^{d-1}$ are still negligible and since Φ is countable, one can take the supremum over entropies $\Phi \in \Phi$ and get the density estimate: for a.e. $x_0 \in J(u)$,

$$c_\Phi(u^+(x_0), u^-(x_0), \nu_u(x_0)) \le \liminf_{r\to 0} \frac{\mu(B(x_0,r))}{\omega_{d-1}r^{d-1}} \quad . \tag{8.2.5}$$

Fourth step: conclusion in the case where Φ is countable Now, (8.2.5) and the Besicovitch derivation Theorem (see Theorem 2.22 in [5]) finally yield

$$c_\Phi(u^+(\cdot), u^-(\cdot), \nu_u(\cdot))\,\mathrm{d}\mathcal{H}^{d-1} \le \mu \quad \text{as measures.}$$

In particular, since $\mu(\overline{\Omega}) \ge \limsup_{\varepsilon\to 0} \mu_\varepsilon(\Omega) \ge \liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon)$ and by definition of E_Φ one gets

$$E_\Phi(u) \le \liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon).$$

Last step: general sets Φ In case where Φ is not countable, we are going to prove that there exists a countable subset $\Phi_0 \subset \Phi$ such that $c_\Phi = c_{\Phi_0}$. More generally, one can prove the following lemma

Lemma 8.2.4. *Let (X, d) be a locally compact metric space. Let $(f_i)_{i\in I}$ be a family of l.s.c. functions on X valued in $\mathbb{R} \cup \{+\infty\}$, where I is any set. Then there exists a countable subset $I_0 \subset I$ such that*

$$\forall x \in X,\ \sup_{i\in I} f_i(x) = \sup_{i\in I_0} f_i(x).$$

Once Lemma 8.2.4 is proved, one can apply it in our situation: $f = c_\Phi : X \to \mathbb{R}^+ \cup \{+\infty\}$, where $X = \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}$. We get a countable subset $\Phi_0 \subset \Phi$ such that $c_\Phi = c_{\Phi_0}$. Then, let us take an increasing sequence $(\Phi_n)_{n\geq 1}$ (for the inclusion) of finite subsets $\Phi_n \subset \Phi_0$ such that $\cup_{n\geq1}\Phi_n = \Phi_0$ so that

$$\forall (u^+, u^-, \nu) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1},\ c_\Phi(u^+, u^-, \nu) = \sup\{c_{\Phi_n}(u^+, u^-, \nu)\ :\ n \geq 1\}.$$

Then (8.2.1) easily follows from the monotone convergence Theorem. Indeed, since the sequence $(c_{\Phi_n})_{n\geq1}$ is non-decreasing, one has

$$\forall u \in \mathrm{BV}(\Omega, \mathbb{R}^d),\ E_\Phi(u) = \sup_{n\geq1} E_{\Phi_n}(u). \quad \square$$

Proof of Lemma 8.2.4. We are going to prove a property slightly more general than Lemma 8.2.4. We claim that for all l.s.c. function $f : X \to \mathbb{R} \cup \{+\infty\}$ such that $f \leq \sup\limits_{i\in I} f_i$, there exists a finite subset $I_0 \subset I$ such that

$$f(x) \leq \sup_{i\in I_0} f_i(x) \quad .$$

One can assume that all the functions f_i, for $i \in I$, and f are real valued. Indeed, given $k \geq 1$, the function $\inf\{f; k\}$ is l.s.c. since for all $\lambda \in \mathbb{R}$, the level set

$$\{x \in X\ :\ \inf\{f; k\} > \lambda\} = \begin{cases} \{x \in X\ :\ f(x) > \lambda\} & \text{if } \lambda < k, \\ \emptyset & \text{if } \lambda \geq k, \end{cases}$$

is an open subset of X. Similarly the functions $\inf\{f_i; k\}$ are l.s.c. for $i \in I$. Then, assuming that there exists a finite subset $I_k \subset I$ such that $\inf\{f; k\} \leq \sup_{i\in I_k} \inf\{f_i; k\}$ and taking the supremum over $k \geq 1$ yield

$$f \leq \sup_{i\in \cup_k I_k} f_i \quad .$$

Moreover, one can assume that f is Lipschitz continuous. Indeed any l.s.c. function is the supremum of a countable family of Lipschitz functions:

$$f = \sup_{k\geq1} f_k \quad \text{where} \quad f_k(x) = \inf_{y\in X} f(y) + k|x - y| \quad \text{for all } x \in X.$$

Similarly, if $f_k \leq \sup_{i\in I_k} f_i$ for some countable set I_k for all $k \geq 1$ then $f \leq \sup_{i\in\cup I_k} f_i$.

Finally, one can assume that X is compact. Indeed, since X is locally compact, there exists a countable family $(X_k)_{k\geq 1}$ of compact subsets such that $X = \cup_k X_k$. If, for each $k \geq 1$, there exists I_k such that $f \leq \sup_{i\in I_k} f_i$ on the set X_k then one can deduce, as above, that $f \leq \sup_{i\in \cup_k I_k} f_i$ on the whole space X. Summing up, it remains to prove the claim when the functions f_i, $i \in I$, and f are real valued, f is Lipschitz continuous and X is compact. Let us fix $\varepsilon > 0$ and $\delta > 0$ such that $\delta L < \varepsilon$ where $L > 0$ is the Lipschitz constant of f. Note that one has the inclusion

$$X \subset \bigcup_{(i,x)\in I\times X \text{ s.t. } f_i(x)\geq f(x)-\varepsilon} \omega_{i,x} \quad \text{where} \quad \omega_{i,x} := \{y \in B(x,\delta) \ : \ f_i(y) > f_i(x) - \varepsilon\}.$$

Indeed, since $f \leq \sup_{i\in I} f_i$, for all $x \in X$ there exists $i \in I$ such that $f_i(x) \geq f(x) - \varepsilon$ and one trivially has $x \in \omega_{i,x}$. Since all functions f_i are l.s.c., the family $(\omega_{i,x})_{(i,x)\in I\times X}$ is an open cover of X. Since X is compact, there exists a finite subcover $(\omega_{i_l,x_l})_{l=1,\ldots,n}$, for some $i_l \in I$, $x_l \in X$, $n \geq 1$, i.e.

$$X \subset \bigcup_{l=1}^{n} \omega_l \quad \text{where} \quad \omega_l := \omega_{i_l,x_l} \quad ,$$

where (i_l, x_l) satisfies $f_{i_l}(x_l) \geq f(x_l) - \varepsilon$. Now for all $y \in X$, there exists $l \in \{1,\ldots,n\}$ such that $y \in \omega_l$. Then, applying the fact that f is Lipschitz continuous with constant $L \in (0, \varepsilon/\delta)$, and the definition of (i_l, x_l) and ω_l, one gets

$$f(y) \leq f(x_l) + L|y - x_l| \leq f_{i_l}(x_l) + \varepsilon + L\delta \leq f_{i_l}(y) + 3\varepsilon \quad .$$

For all $k \geq 1$, applying the preceding for $\varepsilon = \frac{1}{k}$, we get some finite subset $I_k \subset I$ such that

$$\forall y \in X,\ f(y) \leq \sup_{i\in I_k} f_i(y) + \frac{3}{k} \quad .$$

Let $I_0 := \cup_{k\geq 1} I_k$. Taking the lim sup for $k \to \infty$ in the preceding inequality finally yields

$$f \leq \sup_{i\in I_0} f_i \quad ,$$

which is the claim. □

Remark 8.2.5. Note that, in the proof of Theorem 8.2.1, we do not care about the regularity of the limiting configuration u. In general, the regularity of admissible configurations, i.e. limits of finite energy sequences, is a difficult issue. In [27], the authors study the regularity for the Aviles-Giga model. Even in this well known situation, this question turns out to be very hard in particular because admissible structures are not necessarily of bounded variation. The only estimate which is used on admissible vector fields u is the fact that the entropy production, $\nabla \cdot [\Phi(u)]$ is controlled by the energy (as in (8.2.4)) for a large class of entropies $\Phi \in \Phi_0$ (see also [28] for the kind of entropy which is used). In [27], the authors show that any vector field such that the entropy production is finite for all entropy in Φ_0 shares most of the properties of bounded variation functions, in particular the existence of a jump set and traces (see (5.2.2)). In our case, we completely avoid this kind of problems by assuming that $u \in \mathrm{BV}$.

8.3 Applications

The Aviles-Giga model For the Ginzburg-Landau potential, $W(z) = (1-|z|^2)^2$, the entropy set $\Phi := \langle \Phi_0 \rangle$ generated by the single entropy

$$\Phi_0(z) = \left[z_2(1-z_1^2) - \frac{z_2^3}{3} \ ; \ z_1(1-z_2^2) - \frac{z_1^3}{3} \right] \quad \text{for all } z \in \mathbb{R}^2$$

satisfies the saturation condition. This was used in [7] and [39] to prove the following $\Gamma - \liminf$ property:

Theorem 8.3.1 (P. Aviles, Y. Giga)**.** *Let* $\Omega \subset \mathbb{R}^2$ *a bounded subset and* $u \in \mathrm{BV}(\Omega, \mathbb{R}^2)$ *such that* $\nabla \cdot u = 0$. *For every sequence* $(u_\varepsilon)_{\varepsilon>0}$ *converging to* u *in* $L^1(\Omega)$, *one has*

$$E_f(u) \leq \liminf_{\varepsilon \to 0} \frac{1}{2} \int_\Omega \varepsilon |\nabla u_\varepsilon(x)|^2 + \left(1 - |u_\varepsilon(x)|^2\right)^2 \mathrm{d}x \quad ,$$

where $f(u^+, u^-, \nu) = \frac{|u^+ - u^-|^3}{6}$.

Even if Theorem 8.2.1 requires that the sequence $(u_\varepsilon)_{\varepsilon>0}$ converges in $L^3(\Omega)$, one can prove that (8.2.1) still is true if the sequence $(u_\varepsilon)_{\varepsilon>0}$ only converges in L^1 thus implying Theorem 8.3.1. Indeed, let $(u_\varepsilon)_{\varepsilon>0}$ be a bounded energy sequence converging to u in L^1. Since $W(z) \geq C|z|^4$ for z large enough, $(u_\varepsilon)_{\varepsilon>0}$ is bounded in L^4. In particular, $(u_\varepsilon)_{\varepsilon>0}$ is compact in L^3. Indeed,

$$\|u_\varepsilon - u\|_{L^3} \leq \|u_\varepsilon - u\|_{L^4}^\theta \|u_\varepsilon - u\|_{L^1}^{1-\theta} \xrightarrow[\varepsilon \to 0]{} 0,$$

where $\theta \in (0,1)$ is the interpolating exponent: $\frac{1}{3} = \frac{\theta}{4} + \frac{1-\theta}{1}$.

The case where W is the square of a harmonic function As an application of Theorem 8.2.1, we get the following $\Gamma - \liminf$ property:

Theorem 8.3.2. *Let* Ω *be a bounded open subset of* $\mathbb{R}^2$. *We consider the energy* E_ε *defined by* (8.0.1) *in dimension* $d = 2$. *Assume that the potential reads* $W = w^2$ *for some harmonic polynomial* $w : \mathbb{R}^2 \to \mathbb{R}$ *of degree* $q \geq 1$. *Then, for all sequence* $(u_\varepsilon)_{\varepsilon>0}$ *converging in* $L^{q+1}(\Omega)$ *to* $u \in \mathrm{BV}(\Omega, \mathbb{R}^2)$, *one has*

$$E_{1D}(u) \leq \liminf_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon).$$

Proof. Thanks to Theorem 8.2.1 and remark 8.2.3, it is enough to find $\Phi \subset \mathcal{E}(\mathbb{R}^d)$ satisfying $c_W \leq c_\Phi$. The existence of Φ follows from Lemma 8.3.3 below. Indeed, once Lemma 8.3.3 is proved, one can take $\Phi := \{\pm \Phi_\nu \ : \ \nu \in \mathbb{S}^1\}$. □

Lemma 8.3.3. *Assume that* $W = w^2$ *for some harmonic polynomial* $w : \mathbb{R}^2 \to \mathbb{R}$ *of degree* $q \geq 1$. *Then, for all* $\nu \in \mathbb{S}^1$, *there exists* $\Phi_\nu \in \mathcal{E}_W^{q+1}(\mathbb{R}^d)$ *such that*

1. $|[\Phi_\nu(u^+) - \Phi_\nu(u^-)] \cdot \nu| = c_W(u^-, u^+)$ *for every* $u^\pm \in \mathbb{R}^2$ *such that* $\nu \cdot (u^+ - u^-) = 0$ *and* $W(u^\pm) = 0$,

2. there exists a constant $C > 0$ *such that for all* $z \in \mathbb{R}^d$, $|\Phi_\nu(z)| \leq C(1 + |z|^{q+1})$.

Proof. It is enough to prove the lemma when $\nu = e_2$. Indeed assume that Lemma 8.3.3 is proved for $\nu = e_2$. Let us fix $\nu \in \mathbb{S}^1$ and $v^\pm$ such that $\nu \cdot (v^+ - v^-) = 0$. There exists a rotation $\sigma_\nu \in \mathrm{SO}(2)$ such that $\sigma_\nu(e_2) = \nu$. Let us define $W_\nu = W \circ \sigma_\nu^{-1}$ and $u^\pm = \sigma_\nu^{-1}(v^\pm)$. Since the Laplace operator is in invariant by rotation, W_ν still is the square of a harmonic function $w_\nu := w \circ \sigma_\nu^{-1}$. In particular, there exists $\Phi_0 \in \mathcal{E}^{q+1}_{W_\nu}(\mathbb{R}^d)$ satisfying $c_{\{\pm\Phi_0\}}(u^+, u^-, e_2) = c_{W_\nu}(u^-, u^+)$. Then, let us define $\Phi_\nu := \sigma_\nu \cdot \Phi_0$, and let us compute

$$\begin{aligned}
c_{\{\pm\Phi_\nu\}}(\cdot,\cdot,\nu) = c_W(\cdot,\cdot) &= \big|[\Phi_\nu(v^+) - \Phi_\nu(v^-)] \cdot \nu\big| \\
&= \big|[\sigma_\nu \cdot \Phi_\nu(\sigma_\nu(u^+)) - \sigma_\nu \cdot \Phi_\nu(\sigma_\nu(u^-))] \cdot \sigma_\nu(e_2)\big| \\
&= \big|[\Phi_0(u^+) - \Phi_0(u^-)] \cdot e_2\big| \\
&= c_{W_\nu}(u^-, u^+) \\
&= |u^+ - u^-| \textstyle\int_0^1 \sqrt{W \circ \sigma_\nu^{-1}((1-t)u^- + tu^+)}\, \mathrm{d}t \\
&= |v^+ - v^-| \textstyle\int_0^1 \sqrt{W((1-t)v^- + tv^+)}\, \mathrm{d}t \\
&= c_W(v^+, v^-).
\end{aligned}$$

Now, in the case where $\nu = e_2$, Lemma 8.3.3 is a direct consequence of the results of chapter 7. Indeed, thanks to Lemma 7.2.9, there exists $\Phi_0 \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}^2)$ such that the first property of Lemma 8.3.3 is satisfied and for all sequence $(u_\varepsilon)_{\varepsilon>0} \subset \mathcal{C}^\infty \cap L^\infty(\Omega, \mathbb{R}^d)$ such that $E_\varepsilon(u_\varepsilon) + \|u_\varepsilon\|_{L^q(\Omega)}$ is bounded,

$$\begin{aligned}
\nabla \cdot [\Phi_0(u_\varepsilon)] &= w(u_\varepsilon)(\partial_1 u_\varepsilon^2 - \partial_2 u_\varepsilon^1) \\
&\leq \frac{1}{2}\left\{\frac{1}{\varepsilon}\, W(u_\varepsilon) + \varepsilon\left[(\partial_1 u_\varepsilon^2)^2 + (\partial_2 u_\varepsilon^1)^2 - 2\, \partial_1 u_\varepsilon^2\, \partial_2 u_\varepsilon^1\right]\right\} \\
&\leq \frac{1}{2}\left\{\frac{1}{\varepsilon}\, W(u_\varepsilon) + \varepsilon\left[|\nabla u_\varepsilon|^2 + 2(\partial_2 u_\varepsilon^2\, \partial_1 u_\varepsilon^1 - \partial_1 u_\varepsilon^2\, \partial_2 u_\varepsilon^1)\right]\right\} \\
&\leq d_\varepsilon(u_\varepsilon) + \partial_2 u_\varepsilon^2\, \partial_1 u_\varepsilon^1 - \partial_1 u_\varepsilon^2\, \partial_2 u_\varepsilon^1 = d_\varepsilon(u_\varepsilon) + R_\varepsilon \quad ,
\end{aligned}$$

where $R_\varepsilon(\cdot) := \varepsilon \nabla \cdot \{u_\varepsilon^1 [\partial_2 u_\varepsilon^2\, ; -\partial_1 u_\varepsilon^2]\}$. We have to prove that $\Phi_0 \in \mathcal{E}^{q+1}_W(\mathbb{R}^d)$, i.e. Φ_0 is an entropy in the sense of Definition 8.1.1. In order to get (8.1.2), one has to prove that R_ε converges to 0 in the distributional sense. For all $\chi \in \mathcal{D}(\mathbb{R}^2)$, one has

$$\begin{aligned}
\langle R_\varepsilon\, ; \chi\rangle &= \langle u_\varepsilon^1 [\partial_2 u_\varepsilon^2\, ; -\partial_1 u_\varepsilon^2]\, ; \nabla\chi\rangle \\
&\leq \|u_\varepsilon\|_{L^2(\omega)} \|\nabla u_\varepsilon\|_{L^2(\Omega)} \|\nabla\chi\|_{L^\infty(\Omega)}.
\end{aligned}$$

Since $\varepsilon\|\nabla u_\varepsilon\|^2_{L^2(\Omega)} \leq E_\varepsilon(u_\varepsilon) \leq C < +\infty$, one has $\varepsilon\|\nabla u_\varepsilon\|_{L^2(\Omega)} \underset{\varepsilon\to 0}{\longrightarrow} 0$. Moreover, since $q + 1 \geq 2$, the sequence $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $L^2(\Omega)$. In particular, one has $\langle R_\varepsilon\, ; \chi\rangle \underset{\varepsilon\to 0}{\longrightarrow} 0$. It remains to prove the second property, that is the growth condition on Φ_ν, $|\Phi_\nu(z)| \leq C(1 + |z|^{q+1})$: this is a consequence of (7.2.15) and (7.2.16) since w is a polynomial of degree q. □

Remark 8.3.4. In fact each entropy Φ_ν is a harmonic polynomial of degree $q+1$. Indeed, from (7.2.15) one deduces that Φ_ν is harmonic. Then the growth condition on Φ_ν implies that it is a polynomial of degree $q' \leq q+1$. Again, since $\nabla\Phi_\nu$ is of degree $q'-1$, (7.2.15) implies that $q' - 1 = q$.

Exemple 8.3.5. In chapter 7, the potential $W(z) = (z_1 z_2)^2$ was considered. Theorem 8.3.2 applies for sequences $(u_\varepsilon)_{\varepsilon>0}$ strongly converging in L^3.

An example in dimension 3 Let come back to the example of the end of chapter 7:

$$W(z) = (|z|^2 - 1)^2 + 4z_3^2(z_1^2 + z_2^2).$$

Let G be the subgroup composed with all rotations preserving the vertical axis $\mathbb{R}e_3$: $G = \{\sigma \in \mathcal{O}(3) \ : \ \sigma(e_3) = e_3\}$. The potential W and so the entropy set $\mathcal{E}(\mathbb{R}^d)$ is invariant by rotations $\sigma \in G$. Let Φ_0 be an entropy satisfying (7.3.17). In particular Φ_0 is a polynomial of degree 3 and (8.1.1) is satisfied for $q = 3$. Moreover, as in the proof of Theorem 8.3.2, it is easy to see that Φ_0 is an entropy in the sense of Definition 8.1.1. Indeed, by construction of Φ_0 and thanks to (7.2.6) and (7.3.16), for all sequence $(u_\varepsilon)_{\varepsilon>0} \subset \mathcal{C}^\infty \cap L^\infty(\Omega)$ such that $E_\varepsilon(u_\varepsilon) + \|u_\varepsilon\|_{L^q}$ is bounded for some $q \geq 1$,

$$\nabla \cdot [\Phi_0(u_\varepsilon)] \leq d_\varepsilon(u_\varepsilon) + R_\varepsilon \quad ,$$

where takes the form $R_\varepsilon = \varepsilon \sum_{i<j} \partial_i u^j \, \partial_j u^i - \partial_i u^i \, \partial_j u^j$ for $u_\varepsilon = (u_\varepsilon^i)_{i\in\{1,\dots,d\}}$. We prove the convergence of R_ε to 0 in $\mathcal{D}'$ whatever the dimension $d \geq 2$ under the condition $q \geq (2^*)' = \frac{2d}{2+d}$ ($q > 1$ in dimension 2). For all $\chi \in \mathcal{D}(\Omega)$,

$$\begin{aligned}
\langle R_\varepsilon \, ; \chi\rangle &= \varepsilon \sum_{i<j} \langle \partial_i u^j \, \partial_j u^i - \partial_i u^i \, \partial_j u^j \, ; \chi\rangle \\
&\leq \varepsilon \sum_{i<j} \left\langle \nabla \cdot \left\{ u^j [\partial_j u^i \, ; -\partial_i u^i] \right\} \, ; \chi \right\rangle \\
&\leq \varepsilon \sum_{i<j} \left\langle u^j [\partial_j u^i \, ; -\partial_i u^i] \, ; \nabla\chi \right\rangle \\
&\leq C\varepsilon \|u_\varepsilon\|_{L^2} \|\nabla u_\varepsilon\|_{L^2} \|\nabla\chi\|_{L^\infty} \leq C\varepsilon \|u_\varepsilon\|_{L^q}^t \|u_\varepsilon\|_{L^{q'}}^{1-t} \|\nabla u_\varepsilon\|_{L^2} \|\nabla\chi\|_{L^\infty},
\end{aligned}$$

where $t \in (0,1)$ is the interpolating exponent: $\frac{t}{q} + \frac{1-t}{q'} = \frac{1}{2}$. Moreover $\|u_\varepsilon\|_{L^{q'}} \leq C(\|\nabla u_\varepsilon\|_{L^2} + \|u_\varepsilon\|_{L^1})$ since $q' \leq 2^* = \frac{2d}{d-2}$ ($q' < 2^* = \infty$ in dimension 2). Thus, since $\|u_\varepsilon\|_{L^q}$ is bounded,

$$\langle R_\varepsilon \, ; \chi\rangle \leq C\varepsilon (1 + \|\nabla u_\varepsilon\|_{L^2})^{2-t} \xrightarrow[\varepsilon\to 0]{} 0.$$

Then the entropy set $\Phi := G \cdot \Phi_0 = \{\sigma \cdot \Phi_0 \ : \ \sigma \in G\}$ satisfies $c_\Phi \geq c_W$ and Theorem 8.2.1 apply for $q = \frac{2d}{d+2}$ ($q > 1$ in dimension 2). In our exemple, $\Phi_0 \in \mathcal{E}_W^q(\mathbb{R}^3)$ whenever $q \geq \max\{3 \, ; \frac{6}{5}\} = 3$. As we noticed in example the Aviles-Giga case (see example 8.1.10), the $\Gamma - \liminf$ property actually holds for the L^1 convergence: for all sequences $(u_\varepsilon)_{\varepsilon>0}$ converging in L^1 to $u \in \mathrm{BV}(\Omega, \mathbb{R}^3)$, one has

$$E_{1D}(u) \leq \liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon).$$

Conclusion and perspectives

This chapter was mainly concerned with the optimality of the one-dimensional transition layer for Aviles-Giga type energy functionals. In two dimensions, for potentials W vanishing on the circle, we saw that the asymptotic energy should be a free energy defined over unit-length divergence-free vector fields. After investigating these so called "line energies", we found a new necessary condition for the lower semicontinuity of line energies. This condition, based on a very simple geometric construction, avoids the presence of two-dimensional microstructures. Although this excludes a large class of line energies, containing all line energies associated to the cost t^p with $p < 1$, it is far from being sufficient. Yet, it would be very interesting to find a general necessary and sufficient condition (on the cost function) for the lower semicontinuity. However, such a general condition seems hard to establish since every counter-example (as the example of chapter 6) only provides very partial conditions which are not expected to be sufficient. In chapter 7, we discovered a new class of potentials W (containing the Ginzburg-Landau potential $W(z) = (1 - |z|^2)^2$ and some examples in higher dimension) for which the one-dimensional profile is the unique optimal transition layer between two wells of W. In chapter 8, we also deduced a sharp lower bound on these energies corresponding to the same class of potentials. The newest thing was that both the uniqueness of the global minimizer and the fact that our method applied for unusual potentials, namely potentials $W = w^2$ for some harmonic function w. Our main tool was the entropy method which goes back to the pioneering work of P. Aviles, Y. Giga, W. Jin and R. V. Kohn. Although we have only considered divergence-free vector fields in a Euclidean space, it seems that the entropy method, which has some similarities with the calibration method for minimal surfaces, could apply in a more general situation, eventually without divergence constraint. In the case of vector fields defined on a Euclidean space E without divergence constraint, the problem of finding an entropy would come down to find a map $\Phi : E \to E$, such that Φ satisfies the punctual condition $|\nabla\Phi(\cdot)| \leq \sqrt{W(\cdot)}$ and such that this estimate is saturated on the geodesic γ joining u^- to u^+ (so that Φ is determined on γ). As the existence of Lipschitz extension for maps defined on a subspace of some metric space requires some conditions on the curvature (see [43]), one can infer that the question of the optimality of the one-dimensional transition layer is related to the curvature of E endowed with the singular Riemannian metric $g = \sqrt{W} g_0$, where g_0 is the standard Euclidean metric on E.

Appendix A

Minimal length problem in weighted metric spaces

In the first section we remind the main definitions, tools and properties in the theory of length spaces. In the second section, we study the existence of geodesics in a length space endowed with a weight function w vanishing on a non trivial set. In the last section, we investigate the existence of an optimal profile between two wells (or heteroclinic connexion between two phases) in length spaces endowed with some potential W.

A.1 Minimal length problem in metric spaces

Let (X, d) be metric space: X is any set and d is a metric on X.

Length of a curve Given any curve $\gamma : I \to X$ (i.e. a continuous map), where $I \subset \mathbb{R}$ is a non-empty interval, we define the length of γ by the formula

$$L(\gamma) := \sup \sum_{i=0}^{N-1} d(\gamma(t_i), \gamma(t_{i+1})) \in \mathbb{R} \cup \{+\infty\},$$

where the supremum is taken over all $N \geq 1$ and all sequences $t_0 \leq \cdots \leq t_N$ in I. γ is *rectifiable* if $L(\gamma) < \infty$. The length function enjoys the following properties:

Proposition A.1.1. *1. For all curves $\gamma : [a, b] \to X$, one has $L(\gamma) \geq d(\gamma(a), \gamma(b))$.*

2. Let $\gamma_1 : (a, b] \to X$ and $\gamma_2 : [b, c) \to X$ be two curves such that $\gamma_1(b) = \gamma_2(b)$. The concatenation of γ_1 and γ_2, defined on (a, c) by

$$\gamma_1 \cdot \gamma_2(t) = \begin{cases} \gamma_1(t) & \text{if } t \in (a, b], \\ \gamma_2(t) & \text{if } t \in [b, c), \end{cases}$$

satisfies $L(\gamma_1 \cdot \gamma_2) = L(\gamma_1) + L(\gamma_2)$. In particular, for any curve $\gamma : I \to X$ and $J \subset I$, one has $L(\gamma_{|J}) \leq L(\gamma)$.

3. *For any rectifiable curve* $\gamma : I \to X$ *and* $t_0 \in I$*, the mapping* $t \to L(\gamma_{|(t_0,t)})$ *is continuous.*
4. *L is lower semicontinuous for the uniform convergence on compact sets: let* $I \subset \mathbb{R}$ *be an interval and* $(\gamma_n)_{n\geq 1} \subset \mathcal{C}^0(I, X)$ *be a sequence uniformly converging to some curve* $\gamma : I \to X$ *on all compact subsets* $I_0 \subset I$*. Then*

$$L(\gamma) \leq \liminf_{n\to\infty} L(\gamma_n).$$

Proof. The first property comes from the triangle inequality, the second is evident and the last one follows from the l.s.c. of any supremum of l.s.c. functions. The third property is a consequence of the uniform continuity of γ on every compact set. Indeed, one can assume that γ is defined on a compact interval I. Then, given $\varepsilon > 0$, there exists a partition $(t_i)_{i=0,\dots,N}$ such that $a \leq t_0 \leq \cdots \leq t_N \leq b\,,\quad \sum_{i=0}^{N-1} d(\gamma(t_i), \gamma(t_{i+1})) \geq L(\gamma) - \varepsilon$. Since γ is uniformly continuous, up to refine the partition $(t_i)_i$, one can assume that $d(\gamma(t_i), \gamma(t_{i+1}) < \varepsilon$ for all i. Thus, the concatenation property yields

$$L(\gamma) = \sum_{i=0}^{N-1} L(\gamma_{|(t_i,t_{i+1})}) \geq \sum_{i=0}^{N-1} d(\gamma(t_i), \gamma(t_{i+1})) \geq L(\gamma) - \varepsilon \quad .$$

In particular, $\sum_{i=0}^{N-1} L(\gamma_{|(t_i,t_{i+1})}) - d(\gamma(t_{i+1}), \gamma(t_i)) \leq \varepsilon$. Since each term of the sum is nonnegative, one deduces that $L(\gamma_{|(t_i,t_{i+1})}) \leq d(\gamma(t_{i+1}), \gamma(t_i)) + \varepsilon \leq 2\varepsilon$. Thanks to the second property, this implies that the mapping $t \to L(\gamma_{|(t_0,t)})$ is uniformly continuous. □

For Lipschitz maps we have the following representation formula for the length (see [31] for the rectifiability of Lipschitz maps):

Proposition A.1.2. *Let* $\gamma : I \to X$ *be a Lipschitz map. Then, for a.e.* $t \in I$*, the following quantity,*

$$|\dot\gamma|(t) = \lim_{s\to t} \frac{d(\gamma(t), \gamma(s))}{|t-s|} \quad ,$$

is well defined and measurable. $|\dot\gamma|$ *is called* metric derivative *of* γ*. Moreover, one has*

$$L(\gamma) = \int_I |\dot\gamma(t)|\,\mathrm{d}t \quad .$$

In particular, if γ *is rectifiable and* $t_0 \in I$*, then the function* $\varphi : I \to \mathbb{R}$ *defined by* $\varphi(t) = \pm L(\gamma_{|(t_0,t)})$ *for* $\pm(t - t_0) \geq 0$ *belongs to the Sobolev space* $W^{1,1}$ *and*

$$\dot\varphi(t) = |\dot\gamma|(t) \quad a.e.$$

Parameterization If $\gamma : I \to X$ is a curve, and $\varphi : I' \to I$ is a non-decreasing surjective continuous mapping, called parameterization, then the curve $\sigma = \gamma \circ \varphi : I' \to X$ satisfies $L(\sigma) = L(\gamma)$. The curve γ is said to have *constant speed* if for all $t, t' \in I$ s.t. $t < t'$, $L(\gamma_{|(t,t')}) = \lambda|t - t'|$. λ is the *speed* of the curve γ. The curve γ is parameterized by arc length if $\lambda = 1$. Assume that a curve γ satisfies $L(\gamma_{|J}) < \infty$ for all compact subset $J \subset I$. Let us fix $t_0 \in I$ and $\varphi(t) := \pm L(\gamma_{|(t_0,t)})$ for $t \in I$ s.t. $\pm(t - t_0) \geq 0$. φ is continuous, non-decreasing and the curve

$$\sigma : \varphi(I) \to X \ , \quad \sigma(\varphi(t)) = \gamma(t) \tag{A.1.1}$$

is well defined, continuous and parameterized by arc length. Indeed, for $t, t' \in I$ s.t. $t \leq t'$, $\varphi(t') - \varphi(t) = L(\gamma_{|(t,t')}) = L\left(\sigma_{|(\varphi(t),\varphi(t'))}\right)$. Note that γ has constant speed $\lambda \geq 0$ if and only if γ is Lipschitz continuous and $|\dot\gamma(t)| = \lambda$ a.e.

Up to renormalization (using a translation and a homothetie is necessary), it is always possible to consider curves defined on $I = [0, 1]$.

Minimal length problem We define the intrinsic pseudo-metric d_1 by minimizing the length of all curves γ connecting two points $x^\pm \in X$:

$$d_1(x^-, x^+) := \inf\{L(\gamma) \ : \ \gamma \in \mathcal{C}^0(I, X),\ I = [a^-, a^+] \text{ s.t. } \gamma(a^\pm) = x^\pm\}. \tag{A.1.2}$$

Here, if a^+ or a^- is infinite, we use the following convention: $\gamma(\pm\infty) := \lim\limits_{t\to\pm\infty} \gamma(t) = x^\pm$ if it exists. A curve $\gamma \in \mathcal{C}^0([a^-, a^+], X)$ such that $\gamma(a^\pm) = x^\pm$ is called a *path* from a^- to a^+ and we will use the notation $\gamma : x^- \to x^+$.

When (X, d) is a Euclidean space, $d_1 = d$ and the infimum value in (A.1.2) is achieved by the segment $[a^-, a^+]$. In general, the intrinsic pseudo-distance d_1 enjoys the following properties:

Proposition A.1.3. *1. $d_1 : X \times X \to \mathbb{R}^+ \cup \{+\infty\}$ is a pseudo-metric (possibly infinite): d_1 is symmetric, subadditive and $d_1(x, y) = 0$ implies $x = y$. Moreover, $d_1(x, y) \geq d(x, y)$ for all $x, y \in X$.*

2. For all $x, y \in X$, $\varepsilon > 0$ and $t \in (0, 1)$ there exists $z \in X$ such that

$$d_1(x, z) \leq t d_1(x, y) + \varepsilon \quad \textit{and} \quad d_1(z, y) \leq (1 - t) d_1(x, y) + \varepsilon \quad .$$

Remark A.1.4. 1. The second property for $t = 1/2$ (called *middle point property*) is a characterization of intrinsic distances (see Definition A.1.5) in the subclass of metric spaces (X, d) which are complete.

2. $d_1(x, y)$ may be infinite, for instance if x and y belong to two different connected components of X.

Proof. The first claim follows from the properties of the length. For the second claim, one can assume that $d_1(x, y) < \infty$. Let us fix x, $y \in X$, $\varepsilon > 0$ and $t \in (0, 1)$. There exists a path $\gamma : [0, 1] \to X$ joining x and y such that $L(\gamma) \leq d_1(x, y) + \varepsilon$. By continuity, there exists $s \in I$ such that $L(\gamma_{|(0,s)}) = t d(x, y)$. One deduces that $L(\gamma_{|(s,1)}) = L(\gamma) - L(\gamma_{|(0,s)}) \leq (1 - t) d_1(x, y) + \varepsilon$ thus implying the lemma for $z = \gamma(s)$. □

Definition A.1.5. The metric space (X, d) is said to be a *length space* if the distance d is *intrinsic*, i.e. $d = d_1$. (X, d) is a *geodesic space* if it is a length space and any two points $x^\pm$ can be connected by at least one minimal length curve: for all $x^\pm \in X$, there exists a path γ joining x^- and x^+ such that $d(x^-, x^+) = L(\gamma)$.

In a length space, any two points can in particular be connected by at least one rectifiable curve, i.e. (X,d) is *rectifiable arc-connected.*

Proposition A.1.6. *Assume that (X,d) is proper, i.e. every bounded closed subset of (X,d) is compact, and rectifiable arc-connected. Then for any two points $x^\pm$ there exists a length minimizing path joining x^- and x^+. In particular, every length space (X,d) which is proper is a geodesic space.*

Remark A.1.7. A metric space in which any two points can be connected by a minimal length curve is not necessarily a length space since we do not assume that d is intrinsic, i.e. $d = d_1$. It could happen that d and the intrinsic distance d_1 are not even equivalent. Imagine for instance that X is the union of a family of curves $\gamma_n = [0,1] \to X$, $n \geq 1$, such that $\gamma_n(0) = 0$, $\gamma_n(0) =: x_n$ converges to 0, $L(\gamma_n)$ is not bounded, and $\gamma_n((0,1]) \cap \gamma_m((0,1]) = \emptyset$ for $n \neq m$. Then $d(x_n, x) \to 0$ but $d_1(x_n, x) \to \infty$ as $n \to \infty$.

Proof. Let $x^\pm \in X$ be two points. By assumption, there exists at least one curve $\gamma : I = (a^-, a^+) \to X$ such that $\gamma(a^\pm) = x^\pm$ and $L(\gamma) < \infty$. Let $(\gamma_n)_{n\geq1}$ be a sequence of curves connecting x^- and x^+ and minimizing the length: $L(\gamma_n) \longrightarrow d_1(x^-, x^+)$ as $n \to \infty$. Up to renormalization, one can assume that each curve γ_n is defined on $I = [-1/2, 1/2]$ and has constant speed $|\dot{\gamma_n}(t)| =: v_n$. In particular, since $\gamma_n(\pm1/2) = x^\pm$ and $L(\gamma_n) = \int_I |\dot{\gamma_n}| = v_n$ is bounded, the sequence $(\gamma_n)_{n\geq1}$ is equicontinuous and uniformly bounded. Thanks to the Arzelà-Ascoli Theorem, there exists a subsequence of $(\gamma_n)_{n\geq1}$ converging uniformly to some $\gamma \in \mathcal{C}^0(I, X)$. In particular $\gamma(\pm1/2) = x^\pm$ and, due to the l.s.c. of L, γ minimizes the length. ☐

A.2 Minimal length problem in weighted metric spaces

Let (X,d) be metric space and $w : X \to \mathbb{R}^+$ a continuous function called weight function. We say that (X,d,w) is a weighted metric space. $S \subset X$ will denote the set where w vanishes.

w-length of a curve Given any curve $\gamma : I \to X$, we define the w-length of γ by the formula

$$L_w(\gamma) := \sup \sum_{i=0}^{N-1} w(\gamma(t_i)) d(\gamma(t_i), \gamma(t_{i+1})) \in \mathbb{R} \cup \{+\infty\},$$

where the supremum is taken over all $N \geq 1$ and all sequence $t_0 \leq \cdots \leq t_N$ in I. γ is *w-rectifiable* if $L_w(\gamma) < \infty$. The w-length function has the same properties of the length function claimed in Proposition A.1.1 with L_w instead of L except the first claim which must be replaced by

$$L_w(\gamma) \geq w_* L(\gamma) \geq w_* d(\gamma(a), \gamma(b)) \tag{A.2.1}$$

for any curve $\gamma : [a,b] \to X$, where $w_* := \inf\{w(\gamma(t)) \ : \ t \in [a,b]\}$. For Lipschitz maps $\gamma : I \to X$, we have the following representation formula for the w-length

$$L_w(\gamma) = \int_I w(\gamma(t)) |\dot{\gamma}(t)| \, \mathrm{d}t \quad . \tag{A.2.2}$$

In particular, if γ is w-rectifiable and $t_0 \in I$, then the continuous map $\varphi : I \to \mathbb{R}$ defined by $\varphi(t) = \pm L_w(\gamma_{|(t_0,t)})$ for $\pm(t - t_0) \geq 0$ belongs to the Sobolev space $W^{1,1}$ and

$$\varphi'(t) = w(\gamma(t))|\dot{\gamma}|(t) \quad \text{a.e.} \tag{A.2.3}$$

Parameterizations by arc length and w-arc length Assume that $w > 0$ on X. Let γ be a w-rectifiable curve and define $\varphi(t) := \pm L_w(\gamma_{|(t_0,t)})$ for $t \in I$ s.t. $\pm(t - t_0) \geq 0$. φ is continuous, non-decreasing and the curve

$$\sigma : \varphi(I) \to X \ , \quad \sigma(\varphi(t)) = \gamma(t) \tag{A.2.4}$$

is well defined, continuous and *parameterized by w-arc length*, i.e. $L_w(\gamma_{|(t',t)}) = |t' - t|$ for $t, t' \in I$ s.t. $t \leq t'$. Indeed, $\varphi(t') - \varphi(t) = L_w(\gamma_{|(t,t')}) = L_w\left(\sigma_{|(\varphi(t),\varphi(t'))}\right)$. If γ is Lipschitz continuous then γ is parametrized by w-arc length if and only if $w(\gamma(t))|\dot{\gamma}(t)| = 1$ a.e.

Alternatively, one can parameterize the w-rectifiable curve γ by arc-length. Indeed, since w is continuous and positive on X, for all compact subinterval $J \subset I$, $w_* := \inf_{\gamma(J)} w > 0$. In particular, due to (A.2.1), $L(\gamma_{|J}) < \infty$. Then, the parameterization $\varphi(t) := \pm L(\gamma_{|(t_0,t)})$ for $\pm(t - t_0) \geq 0$ with $t_0 \in I$ fixed, is non decreasing and continuous. In particular, the curve $\sigma : \varphi(I) \to X \ , \quad \sigma(\varphi(t)) = \gamma(t)$ is well defined, continuous and parameterized by arc length. Note that if $\inf_{\gamma(I)} w = 0$, γ is non necessarily rectifiable and $\varphi(I) = \mathbb{R}$ whenever $L(\gamma) = \infty$.

Remark A.2.1. Even if $w > 0$ on X, a w-rectifiable curve is not necessarily rectifiable. Indeed, assume that $X = \mathrm{Im}(\gamma)$ where $\gamma : \mathbb{R}^+ \to \mathbb{R}^2$ is a Lipschitz injective curve such that $L(\gamma) = +\infty$ and $\mathrm{Im}(\gamma)$ is closed. Then γ is w-rectifiable whenever $w : X \to \mathbb{R}^+$ is defined by $w(\gamma(t)) = f(t)$ for some integrable function $f : \mathbb{R}^+ \to \mathbb{R}^+$.

Intrinsic distance in weighted metric spaces We define the *intrinsic pseudo-metric d_w associated to the weight w* by minimizing the length of all curves γ connecting two points $x^\pm \in X$:

$$d_w(x^-, x^+) := \inf\{L_w(\gamma) \ : \ \gamma \in \mathcal{C}^0(I, X), \, I = [a^-, a^+] \text{ s.t. } \gamma(a^\pm) = x^\pm\}. \tag{A.2.5}$$

When $w \equiv 1$, $d_w = d_1$ matches with the classical intrinsic metric of (X, d). In general, d_w has the same properties that d_1 except that it is not reflexive if w vanishes on a non trivial w-rectifiable curve:

Proposition A.2.2.

1. *$d_w : X \times X \to \mathbb{R}^+ \cup \{+\infty\}$ is a pseudo-metric (possibly infinite and non reflexive): d_w is symmetric and subadditive. Moreover, the restriction of d_w to the subset $S^c = \{x \in X \ : \ w(x) \neq 0\}$ is a metric on S^c.*
2. *For all $x, y \in X$, $\varepsilon > 0$ and $t \in (0, 1)$ there exists $z \in X$ such that*

$$d(x, z) \leq t d(x, y) + \varepsilon \quad \text{and} \quad d(z, y) \leq (1 - t) d(x, y) + \varepsilon \quad .$$

Proof. The proof is the same that the proof of Proposition A.1.3. □

We are interesting in the existence of a curve $\gamma : x^- \to x^+$ which achieves the minimum in (A.2.5), that is $L_w(\gamma) = d_w(x^-, x^+)$. It is not difficult to show the existence of a w-length minimizing curve if (X, d) is proper, rectifiable arc-connected and $w_* := \inf_{x\in X} w(x) > 0$. Indeed, let $(\gamma_n)_{n\geq 1}$ be a w-length minimizing sequence of curves defined on $I = [0, 1]$ and parameterized in such a way that each curve has constant speed $L(\gamma_n)$. Then, $L_w(\gamma_n) \geq w_* L(\gamma_n)$ and the proof of Proposition A.1.6 shows that there exists a subsequence uniformly converging to a curve $\gamma : x^- \to x^+$ on compact sets. Thanks to the l.s.c. of L_w, this implies that γ is a w-length minimizing path. By contrast, if $\inf_{x\in X} w(x) = 0$, a minimal length curve may not exist:

Exemple A.2.3. Let $S^\pm := \{(x, y) \in \mathbb{R}^2 \ : \ x > 0 \text{ and } \pm\, y > \frac{1}{x}\}$, $S := S^+ \cup S^-$ and $w : X := \{(x, y) \in \mathbb{R}^2 \ : \ x > 0\} \to \mathbb{R}^+$ defined by $w(x) = d(x, S)$ for the Euclidean distance d. Then $d_w(x^-, x^+) = 0$ for $x^\pm \in S$ and the infimum value in (A.1.2) is not achieved whenever $u^+ \in S^+$ and $u^- \in S^-$. Indeed any curve γ connecting u^- and u^+ has to pass through the x-axis at some time. In particular $L_w(\gamma) > 0$. However, one can choose a curve γ starting at u^-, going far away in the direction of the x-axis, then going through the x-axis and coming back to u^+. In such a way, one can make $L_w(\gamma)$ as small as possible.

Existence of geodesics when the singular set S is discrete From now on, we make the following assumptions on (X, d):

$$(X, d) \text{ is a length space and } (X, d) \text{ is proper.}$$

In particular, thanks to Proposition A.1.6, (X, d) is a geodesic space. When the singular set $S := \{x \in X \ : \ w(x) = 0\}$ is not trivial, the existence of geodesics for the length L_w is not evident. We need the following additional assumptions on w and S which will insure the existence of geodesics:

$$\textbf{(H)} \begin{cases} 1. \ \ S \subset X \text{ is a discrete set.} \\ 2. \ \ \omega(\varepsilon) := \inf\{w(x) \ : \ d(x, S) > \varepsilon\} > 0 \text{ for all } \varepsilon > 0. \\ 3. \ \ \text{For all } x_0 \in X,\ d_w(x_0, x) \text{ tends to } +\infty \text{ with } d(x_0, x). \end{cases} \tag{A.2.6}$$

Since (X, d) is proper, the first condition is equivalent to say that for all compact subset $Y \subset X$, $Y \cap S$ is finite. The following proposition provides some examples where all these conditions are fulfilled.

Proposition A.2.4. *Assume that that one of the following condition is fulfilled:*

(H1) *S is finite and (X, d) is compact.*

(H2) *S is finite and there exists a compact set $Y \subset X$ such that $w_* := \inf_{x\in Y^c} w(x) > 0$.*

(H3) *$d_* := \inf\{d(x, y) \ : \ x, y \in S \text{ s.t. } x \neq y\} > 0$ and there exists a compact set $Y \subset X$ such that $w_* := \inf_{x\in Y^c} w(x) > 0$.*

*Then the hypothesis **(H)** is satisfied.*

Proof. The implications **(H1)** $\Rightarrow$ **(H2)** $\Rightarrow$ **(H3)** are quite straightforward. It remains to prove the implication **(H3)** $\Rightarrow$ **(H)**. If the condition **(H3)** is fulfilled, then the first two conditions of **(H)** are satisfied. Let us prove the third condition, $d_w(x_0, x)$ tends to

$+\infty$ with $d(x_0, x)$. We first prove that there exists a positive constant c_* depending on d_* such that

$$\forall x, y \in X,\ d(x, y) \geq d_* \Rightarrow d_w(x, y) \geq c_* \quad .$$

Let S_* be the disjoint union $S_* := \cup_{z\in S} B(z, \frac{d_*}{4})$ and $x, y \in X$ such that $d(x, y) \geq d_*$. If x or y belongs to S_*, say $x \in B(z, \frac{d_*}{4})$ for $z \in S$, then $y \notin B(z, \frac{d_*}{2})$ and $d_w(x, y) \geq \frac{d_*}{4}\omega(\frac{d_*}{4}) > 0$. Indeed, any path $\gamma : x \to y$ has to connect $B(z, \frac{d_*}{4})$ to $B^c(z, \frac{d_*}{2})$. If $x, y \notin S_*$, let $\gamma : [0, 1] \to X$ be a w-rectifiable path joining x and y. If $\mathrm{Im}(\gamma) \cap S'_* = \emptyset$, where $S'_* := \cup_{z\in S} B(z, \frac{d_*}{8})$, then $L_w(\gamma) \geq \omega(\frac{d_*}{8})L(\gamma) \geq \omega(\frac{d_*}{8})d_* > 0$. Otherwise there exists $z \in S$ and $t \in (0, 1)$ such that $\gamma(t) \in B(z, \frac{d_*}{8})$. Then $L_w(\gamma) \geq L_w(\gamma_{|(0,t)}) \geq \omega(\frac{d_*}{8})\frac{d_*}{8}$ since the path $\gamma_{|(0,t)}$ connects $\gamma(0) = x \in B^c(z, \frac{d_*}{4})$ to $\gamma(t) \in B(z, \frac{d_*}{8})$.

We now prove that there exists a constant $c_{**} > 0$ depending on d_* (one can choose $c_{**} = \frac{c_*}{2d_*}$), such that

$$\forall x, y \in X,\ d(x, y) \geq d_* \Rightarrow d_w(x, y) \geq c_{**}\, d(x, y),$$

which is stronger than the third assumption of **(H)**. Let $x, y \in X$ be two points such that $d(x, y) \geq d_*$ and take $N \geq 1$ such that $Nd_* \leq d(x, y) \leq (N + 1)d_*$. Let $\gamma : [0, 1] \to X$ be a path joining x to y. By continuity of γ, there exists a sequence $(t_i)_{i=1,\dots,N}$ such that $0 \leq t_1 \leq \dots \leq t_N \leq 1$ and $\gamma(t_i) \in \partial B(x, id_*)$ for $i = 1, \dots, N$. Then $L_w(\gamma) \geq \sum_{i=1}^{N} L_w(\gamma_{|(t_i,t_{i+1})}) \geq \sum_{i=1}^{N} d_w(\gamma(t_i), \gamma(t_{i+1})) \geq Nc_*$. Taking the infimum over all paths $\gamma : x \to y$, one gets

$$d_w(x, y) \geq Nc_* = \frac{Nc_*(N + 1)d_*}{(N + 1)d_*} \geq \frac{c_*}{2d_*} d(x, y). \quad \square$$

Theorem A.2.5. *Let (X, d) be a length space which is proper and $w : X \to \mathbb{R}^+$ a continuous function satisfying **(H)**. Then d_w is a metric equivalent to d, i.e. d_w induces on X the same topology as d. Moreover, a set $Y \subset X$ is bounded for d_w if and only if Y is bounded for d. In particular (X, d_w) is proper.*

Moreover the length function associated to the metric d_w on X is L_w. In particular (X, d_w) is a length space and, since (X, d_w) is proper, (X, d_w) is a geodesic space.

Proof. We start by proving that d_w is a metric on X. First of all, d_w is finite on $X \times X$. Indeed, since (X, d) is a length space, any two points $x, y \in X$ can be connected by a rectifiable curve $\gamma : [0, 1] \to X$. Since w is continuous, w is bounded on the compact set $\gamma([0, 1])$ and we deduce that γ is also w-rectifiable so that $d_w(x, y) \leq L_w(\gamma) < +\infty$. Due to Proposition A.2.2, it remains to prove that $d_w(x, y) = 0$ implies that $x = y$. Let $x, y \in X$ be two points with $d_w(x, y) = 0$ and let us take a sequence of curves $\gamma_n : [0, 1] \to X$, $n \geq 1$, connecting x to y and such that $L_w(\gamma_n) \longrightarrow 0$ as $n \to +\infty$. The third property in the assumption **(H)** and the fact that (X, d) is proper imply that there exists a compact set $Y \subset X$ containing the image of all curves γ_n. Since S is discrete, $Y \cap S$ is finite and, assuming that $x \neq y$, one can find $\varepsilon > 0$ such that $(S \cup \{y\}) \cap B(x, 3\varepsilon) \subset \{x\}$. As before, since each curve γ_n has to cross the ring $\mathcal{C}_\varepsilon := B(x, 2\varepsilon) \setminus B(x, \varepsilon)$ on which one has $w(x) \geq \omega(\varepsilon)$, one has $L_w(\gamma_n) \geq \varepsilon\omega(\varepsilon)$. This is a contradiction with the fact that $L_w(\gamma_n) \longrightarrow 0$. Thus $x = y$.

The fact that bounded sets for d_w are bounded for d is a direct consequence of the third assumption in **(H)**. Moreover, it is clear from the continuity of w that bounded sets for d are bounded for d_w. More precisely, $d_w(x,y) \leq Cd(x,y)$ for any x, y in a bounded subset $Y \subset X$ (for the metric d), where the constant C depends on Y. This implies that d induces a stronger topology than d_w.

It remains to prove that d_w induces a stronger topology than d. Since bounded sets for d_w and d are the same and since (X,d) is proper, it is enough to prove this claim when (X,d) is compact. Let $(x_n)_{n\geq 1} \subset X$ be a sequence converging to $x \in X$ for d_w. Let y be a limit point of $(x_n)_{n\geq 1}$ in (X,d). Then y is also a limit point of $(x_n)_{n\geq 1}$ in (X,d_w) and so $y = x$. Since X is compact, this implies that $(x_n)_{n\geq 1}$ converges to x in (X,d).

The fact that (X,d_w) is a length space follows from the definition of d_w. Actually, we shall prove that L_w is the length function associated to the metric d_w and so d_w (which is defined by minimizing L_w) is the intrinsic metric of (X,d_w). Let L'_w be the length function in (X,d_w). We have to prove that $L'_w = L_w$. By definition, this corresponds to prove that for any curve $\gamma : I \to X$,

$$\sup \sum_{i=0}^{N-1} d_w(\gamma(t_i),\gamma(t_{i+1})) = \sup \sum_{i=0}^{N-1} w(\gamma(s_i))d(\gamma(s_i),\gamma(s_{i+1})),$$

where both supremums are taken over all $N \geq 1$ and all sequences $t_0 \leq \cdots \leq t_N$ and $s_0 \leq \cdots \leq s_N$ in I. If I is compact, this property easily follows from the uniform continuity of w on $\mathrm{Im}(\gamma)$ and the inequality

$$\inf_{t\in(t_i,t_{i+1})} w(\gamma(t))\ d(\gamma(t_i),\gamma(t_{i+1})) \leq d_w(\gamma(t_i),\gamma(t_{i+1})) \leq \sup_{t\in(t_i,t_{i+1})} w(\gamma(t))\ d(\gamma(t_i),\gamma(t_{i+1})).$$

If I is not compact, one can remark that $L_w(\gamma) = \sup_{J\subset I} L_w(\gamma_{|J})$ where the supremum is taken over compact subintervals $J \subset I$. Since the same property holds for L'_w instead of L_w, one deduces that $L_w = L'_w$. □

Corollary A.2.6. *Let (X,d) be a length space which is proper and $w : X \to \mathbb{R}^+$ a continuous function satisfying **(H)**. For every $x, y \in X$, there exists a path $\gamma : x \to y$ such that*

$$L_w(\gamma) = d_w(x,y).$$

Proof. Indeed, thanks to Theorem A.2.5, (X,d_w) is a geodesic space admitting L_w as length function. □

Remark A.2.7.

1. The assumption "(X,d) is proper" is essential. If one replaces it by the weaker condition "(X,d) is locally compact" then the proposition is false. One can consider for instance $X = \{(x,y) \in \mathbb{R}^2 \ :\ 0 < y\}$ endowed with the weight function $w((x,y)) = 1 + |y|$. $d_w((-1,1),(1,1)) = 1/2$ is not achieved in X but achieved in $\overline{X}$ by the path $[(-1,1),(-1,0)] \cup [(-1,0),(1,0)] \cup [(1,0),(1,1)]$.
2. Although the assumption **(H)** does not seem to be sharp, it cannot be removed as shown in Example A.2.3. Note that, discretizing the set S of this example could also provide a counter-example with a singular set S which is discrete.

A.3 Optimal profile in metric spaces

Let (X, d) be a geodesic space which is proper and $w : X \to \mathbb{R}^+$ a continuous function satisfying **(H)**. In particular, the weighted distance d_w is a metric and (X, d_w) is a geodesic space. Let us consider the potential $W : X \to \mathbb{R}^+$ defined by $W(x) = w^2(x)$ for all $x \in X$. Let us consider the following energy,

$$E_W(\gamma) = \frac{1}{2} \int_{\mathbb{R}} |\dot{\gamma}|^2(t) + W(\gamma(t))\, \mathrm{d}t \quad , \tag{A.3.1}$$

defined over locally Lipschitz curves $\gamma : \mathbb{R} \to X$ such that $\lim\limits_{t\to\pm\infty} \gamma(t)$ exists: such a curve will be called *admissible curve* in all this section. For any locally Lipschitz curve $\gamma : I \to X$, $I \subset \mathbb{R}$ and for all subset $J \subset I$, we define the energy of γ on J by

$$E_W(\gamma, J) = \frac{1}{2} \int_J |\dot{\gamma}|^2(t) + W(\gamma(t))\, \mathrm{d}t \quad . \tag{A.3.2}$$

Note that, if $I = [a, b]$ for $a, b \in \mathbb{R}$ and $W(\gamma(a)) = W(\gamma(b)) = 0$, $E_W(\gamma, I) = E_W(\gamma_I)$ where $\gamma_I : \mathbb{R} \to X$ is defined by

$$\gamma_I(t) = \begin{cases} \gamma(t) & \text{if } t \in [a, b], \\ \gamma(a) & \text{if } t < a, \\ \gamma(b) & \text{if } t > b. \end{cases}$$

Given $x^-, x^+ \in X$, we want to minimize E_W over all admissible curves $\gamma : x^- \to x^+$, i.e. s.t. $\gamma(\pm\infty) = x^\pm$. The minimal energy is denoted by c_W:

$$c_W(x, y) := \inf\{E_W(\gamma) \ : \ \gamma \in \mathrm{Lip}_{loc}(\mathbb{R}, X),\, \gamma : x^- \to x^+\}. \tag{A.3.3}$$

c_W is the *cost function* associated to the potential W. Let us start by a few observations on the cost function c_W.

Proposition A.3.1. *1. E_W is invariant by translation: $E_W(\gamma_{t_0}) = E_W(\gamma)$ for all admissible curves γ, where $\gamma_{t_0}(t) = \gamma(t - t_0)$.*

2. For all $\gamma \in \mathrm{Lip}_{loc}(\mathbb{R}, X)$, $E_W(\gamma) \geq L_w(\gamma)$ and $E_W(\gamma) = L_w(\gamma)$ whenever $|\dot{\gamma}|^2(t) = W(\gamma(t))$ a.e. In particular $c_W(x^-, x^+) \geq d_w(x^-, x^+)$ for all $x^-, x^+ \in X$.

3. For all $x^-, x^+ \in X$, $c_W(x^-, x^+) < \infty \iff W(x^-) = W(x^+) = 0$.

4. c_W is a pseudo metric on X: it satisfies all the properties of a metric except the finiteness on $X \times X$.

The lack of compactness, due to the invariance by translation of the energy E_W, is the reason why the study of the existence of a minimizer in (A.3.3) is not evident from a variational argument.

Proof. The first claim follows from the invariance of the Lebesgue measure by translation and the inequality $E_W \geq L_w$ follows from the Young inequality, $ab \leq \frac{1}{2}(a^2 + b^2)$.

Let us prove the third statement. If $c_W(x^-, x^+) < \infty$, there exists a locally Lipschitz curve $\gamma : x^- \to x^+$ such that $\int_{\mathbb{R}} W(\gamma(t))\, \mathrm{d}t < \infty$. Since, $W(\gamma(t))$ tends to $W(x^\pm)$ as

$t \to \pm\infty$, one has $W(x^\pm) = 0$. Conversely, if $W(x^\pm) = 0$, let $\gamma : [0, L] \to X$ be geodesic in (X, d) parameterized by arc length (so that $L = L(\gamma)$) and such that $\gamma : x^- \to x^+$. One can extend γ on $\mathbb{R}$: we set $\gamma(t) = x^-$ for $t < 0$ and $\gamma(t) = x^+$ for $t > L$. Then,

$$E_W(\gamma) = \frac{1}{2}\int_0^L |\dot\gamma|^2 + W(\gamma) < +\infty \quad .$$

To finish, let us prove that $c_W : X \times X \to \mathbb{R}^+ \cup \{+\infty\}$ shares the axioms of a distance: coincidence axiom, symmetry and subadditivity. First of all c_W is symmetric since $E_W(\gamma) = E_W(\overline{\gamma})$ for any admissible curve $\gamma : x^- \to x^+$, where $\overline{\gamma}(t) := \gamma(-t)$. Moreover, $c_W(x^-, x^+) = 0$ implies $d_w(x^-, x^+) = 0$ and so $x^- = x^+$ since d_w is a distance. It remains to prove the subadditivity. Given $x, y, z \in X$, we shall prove that $c_W(x, y) \leq c_W(x, z) + c_W(z, y)$. One can assume that $c_W(x, z)$, $c_W(z, y) < \infty$. Let us fix $\varepsilon \in (0, 1)$ and let $\gamma_{xz} : x \to z$ (resp. $\gamma_{zy} : z \to y$) be an admissible curve such that $E_W(\gamma_{xz}) \leq c_W(x, z) + \varepsilon$ (resp. $E_W(\gamma_{zy}) \leq c_W(z, y) + \varepsilon$). Let us take $R > 0$ big enough so that

$$\begin{aligned} &E_W(\gamma_{xz}, (-\infty, R]) \geq E_W(\gamma_{xz}) - \varepsilon \quad , \quad d(\gamma_{xz}(R), z) \leq \varepsilon \quad \text{and} \\ &E_W(\gamma_{zy}, [-R, +\infty)) \geq E_W(\gamma_{zy}) - \varepsilon \quad , \quad d(\gamma_{zy}(-R), z) \leq \varepsilon \quad . \end{aligned}$$

Let us pick a geodesic $\gamma : \gamma_{xz}(R) \to \gamma_{zy}(-R)$ parameterized by arc length in (X, d). Up translation, one can assume that γ is defined on $(-L/2, L/2)$ where $L := L(\gamma) = d(\gamma_{xz}(R), \gamma_{zy}(-R)) < 2\varepsilon$. In particular,

$$E_W(\gamma, [-L/2, L/2]) \leq \frac{L}{2}\left(1 + \sup W(\gamma(t))\right) \leq \varepsilon(1 + W^*),$$

where $W^* = \sup\{W(x) \ : \ x \in \overline{B}(z, 3)\}$. Here, we used the fact that $\mathrm{Im}(\gamma) \subset \overline{B}(z, 3)$. Indeed, for all $t \in (-L/2, L/2)$, $d(\gamma(t), z) \leq d(\gamma(t), \gamma_{xz}(R)) + d(\gamma_{xz}(R), z) \leq L + \varepsilon \leq 3\varepsilon \leq 3$. Finally, one can define an admissible curve $\gamma_{xy} : x \to y$ by

$$\gamma_{xy}(t) = \begin{cases} \gamma_{xz}(t + R + \frac{L}{2}) & \text{if } t \leq -\frac{L}{2}, \\ \gamma(t) & \text{if } -\frac{L}{2} < t < \frac{L}{2}, \\ \gamma_{zy}(t - R - \frac{L}{2}) & \text{if } t \geq \frac{L}{2}. \end{cases}$$

One has

$$\begin{aligned} c_W(x, y) &\leq E_W(\gamma_{xy}) \\ &\leq E_W(\gamma_{xz}, (-\infty, R]) + E_W(\gamma, (-L/2, L/2)) + E_W(\gamma_{zy}, [-R, +\infty)) \\ &\leq E_W(\gamma_{xz}) + E_W(\gamma_{zy}) + (3 + W^*)\varepsilon \\ &\leq c_W(x, z) + c_W(z, y) + (5 + W^*)\varepsilon \quad . \end{aligned}$$

Applying this inequality for ε arbitrary small finally yields the desired inequality. □

We know investigate the existence of minimizers for (A.3.3).

Proposition A.3.2. *1. c_W and d_w match on the singular set $S = \{W = 0\}$: $c_W(x^-, x^+) = d_w(x^-, x^+)$ whenever $W(x^\pm) = 0$.*

2. *Let $x^\pm \in S$ be two wells such that there exists a geodesic $\gamma_0 : [a^-, a^+] \to X$ in (X, d_w) with $\gamma(a^\pm) = x^\pm$ and $w(\gamma_0(t)) > 0$ for $t \in (a^-, a^+)$. Then there exists an admissible curve $\gamma : x^- \to x^+$ (obtained by renormalization of γ_0) such that $E_W(\gamma) = c_W(x^-, x^+)$. In particular the minimization problem* (A.3.3) *has a solution.*

Proof. We first prove the second statement. Thanks to the second claim of Proposition A.3.1, it is enough to construct an admissible curve $\gamma : x^- \to x^+$ such that $|\dot\gamma|^2(t) = W(\gamma(t))$ a.e. Let $\gamma_0 : [a^-, a^+] \to X$ be a curve satisfying the assumptions of Proposition A.3.2. Since $w \circ \gamma_0(s) > 0$ on the open set $(-a^-, a^+)$, γ_0 is locally rectifiable (for the metric d) inside $(-a^-, a^+)$: for all compact subset $J \subset [a^-, a^+]$, $L((\gamma_0)_{|J}) < \infty$ (see (A.2.1)). In particular, the parameterization of γ_0 by arc length (for the metric d) is well defined (note that γ_0 is a geodesic in $(X, d_w) \neq (X, d)$). Thus, up to renormalization, we get a curve $\gamma_0 : [-L/2, L/2] \to X$ such that

$$|\dot{\gamma_0}| = 1 \text{ a.e.}, \; w(\gamma_0(t)) > 0 \text{ on } (-L/2, L/2), \; \gamma_0(\pm L/2) = x^\pm \text{ and } L_w(\gamma_0) = d_w(x^-, x^+).$$

Here $L = L(\gamma_0) \in [0, +\infty)$. We look for an admissible curve $\gamma : \mathbb{R} \to X$ of the form

$$\gamma(t) = \gamma_0(\varphi(t)),$$

where $\varphi : \mathbb{R} \to (-L/2, L/2)$ is $\mathcal{C}^1$, increasing and surjective. For γ to satisfy the equipartition condition, i.e. $|\dot\gamma|(t) = w(\gamma(t))$ a.e., we ask φ to be solution of the ODE,

$$\varphi'(t) = F(\varphi(t)), \tag{A.3.4}$$

where $F : \mathbb{R} \to \mathbb{R}$ is the continuous function defined by $F(y) = w(\gamma_0(y))$ for $y \in (-L/2, L/2)$ and $F(y) = 0$ for $\pm y \geq L/2$. Thanks to the Peano-Arzelà theorem, (A.3.4) admits at least one maximal solution $\varphi_0 : I = (a^-, a^+) \to \mathbb{R}$ such that $0 \in I$ and $\varphi_0(0) = 0$. Since F vanishes out of $(-L/2, L/2)$, we know that $\mathrm{Im}(\varphi_0) \subset [-L/2, L/2]$. Moreover, since φ_0 is non decreasing on I, it converges to two distinct stationary points (otherwise φ_0 would be constant) of the preceding ODE. As $F > 0$ on $(-L/2, L/2)$, we have $\lim_{t \to a^\pm} \varphi_0(t) = \pm L/2$. We deduce that φ_0 is an entire solution of the preceding ODE, i.e. $I = \mathbb{R}$. Indeed, if $I \neq \emptyset$, say $a^+ < +\infty$, then one could extend φ_0 by setting $\varphi_0(t) = L/2$ for $t > a^+$. Finally, the curve $\gamma := \gamma_0 \circ \varphi_0$ satisfies $\gamma(\pm\infty) = x^\pm$, $|\dot\gamma|(t) = w(\gamma(t))$ a.e. and so

$$E_W(\gamma) = L_w(\gamma) = L_w(\gamma_0) = d_w(x^-, x^+) \leq c_W(x^-, x^+).$$

Thus, γ minimizes E_W over all admissible connexions between x^- and x^+.

We are now able to prove the first claim of Proposition A.3.2. Let $\gamma : x^- \to x^+$ be a geodesic in (X, d_w) parameterized by L_w-arc length. In particular $\gamma : [a^-, a^+] \to X$ is injective. Since S is discrete and $\mathrm{Im}(\gamma)$ is compact, $S \cap \mathrm{Im}(\gamma)$ is finite. Let $t_1 = a^- < t_2 < \cdots < t_n = a^+$, $i = 1, \ldots, n$, be the sequence of instants for which $\gamma(t_i) \in S$. Thanks to the second claim of Proposition A.3.2, one has $d_w(\gamma(t_i), \gamma(t_{i+1})) = c_W(\gamma(t_i), \gamma(t_{i+1}))$ for $i \in \{1, \ldots, n-1\}$. Hence,

$$c_W(x^-, x^+) \leq \sum_{i=1}^{n-1} c_W(\gamma(t_i), \gamma(t_{i+1})) = \sum_{i=1}^{n-1} d_w(\gamma(t_i), \gamma(t_{i+1})) = d_w(x^-, x^+),$$

which implies the first claim of Proposition A.3.2. ☐

Corollary A.3.3. *Assume that* (X, d_w) *is geodesic and that* d_w *satisfies the "strict triangle inequality" on the set* S*, i.e.* $d_w(x, y) < d_w(x, z)+d_w(z, y)$ *whenever* x, y, z *are distinct points in* S*. Then, for every* $x^\pm \in S$*, there exists an admissible curve* $\gamma : x^- \to x^+$ *such that* $E_W(\gamma) = c_W(x^-, x^+)$*.*

Proof. Thanks to Proposition A.3.2, it is enough to find a geodesic $\gamma : x^- \to x^+$ that meets S only at its end-points. This is a consequence of the "strict triangle inequality". Indeed, let $\gamma : [a^-, a^+] \to X$ be any geodesic (for the metric d_w) connecting two distinct points x^- and x^+ in S. If there was an instant $t \in (a^-, a^+)$ such that $z := \gamma(t) \in S$, we would have $d_w(x^-, x^+) = L_w(\gamma) = L_w(\gamma_{|(a^-,t)}) + L_w(\gamma_{|(t,a^+)}) = d_w(x^-, z) + d_w(z, x^+)$. This is a contradiction with the strict triangle inequality since γ is injective so that x^-, z, x^+ are distinct. □

In view of Proposition A.3.2, the existence of a solution for (A.3.3) seems to require that there exists a geodesic which meets S only at its end-points. We are going to prove that this is actually a necessary condition for the existence of a minimizer when w is Lipschitz. In order to prove this claim, we need the following fundamental lemma (which is true for any continuous potential W):

Lemma A.3.4. *Let* $x^\pm \in S$*. Assume that* $\gamma : x^- \to x^+$ *is a global minimizer of* (A.3.3)*, i.e.* γ *is admissible and* $E_W(\gamma) = c_W(x^-, x^+)$*. Then* γ *satisfies the equipartition identity:*

$$|\dot\gamma|^2(t) = W(\gamma(t)) \quad \textit{a.e. in } \mathbb{R}. \tag{A.3.5}$$

In particular $E_W(\gamma) = L_w(\gamma)$*.*

Proof. Since a.e. point in $\mathbb{R}$ is a Lebesgue point for $t \to |\dot\gamma|^2(t)$ and $t \to W(\gamma(t))$, it is enough to prove that for all open non empty interval $I \subset \mathbb{R}$,

$$\int_I \left\{|\dot\gamma|^2(t) - W(\gamma(t))\right\} \mathrm{d}t = 0.$$

Let $t_0 \in \mathbb{R}$ and $r > 0$ such that $I = (t_0 - r, t_0 + r)$ and define the admissible curve $\gamma_\lambda : \mathbb{R} \to X$ by

$$\gamma_\lambda(t) = \begin{cases} \gamma(t_0 + \lambda t) & \text{if } -\frac{r}{\lambda} < t < \frac{r}{\lambda}, \\ \gamma(t + t_0 + r(1 - 1/\lambda)) & \text{if } t \ge \frac{r}{\lambda}, \\ \gamma(t + t_0 - r(1 - 1/\lambda)) & \text{if } t \le -\frac{r}{\lambda}. \end{cases}$$

Then, the affine change of variable $s = t_0 + \lambda t$ on I yields

$$E_W(\gamma_\lambda) = E_W(\gamma, I^c) + \frac{1}{2}\int_I \lambda|\dot\gamma|^2(s) + \frac{1}{\lambda}W(\gamma(s))\,\mathrm{d}s \quad .$$

Note that $\gamma_1 = \gamma$ and, by optimality of γ, $E_W(\gamma_\lambda) \ge E_W(\gamma)$ for all $\lambda > 0$. Thus,

$$0 = \frac{\mathrm{d}}{\mathrm{d}\lambda}E_W(\gamma_\lambda)\Big|_{\lambda=1} = \frac{1}{2}\int_I \left\{|\dot\gamma|^2(t) - W(\gamma(t))\right\} \mathrm{d}t \quad . \quad \square$$

Proposition A.3.5. *Assume that* $w : (X, d) \to \mathbb{R}^+$ *is locally Lipschitz continuous. Let us take* $x^\pm \in S$ *such that all geodesics* $\gamma : x^- \to x^+$ *in* (X, d_w) *satisfy* $\mathrm{Im}(\gamma) \cap S \setminus \{x^\pm\} \neq \emptyset$. *In particular,* $x^- \neq x^+$.

Then, for all admissible curve $\gamma : x^- \to x^+$, $E_W(\gamma) > c_W(x^-, x^+)$. *In other words, the minimization problem* (A.3.3) *has no solution.*

Remark A.3.6. The Lipschitz condition on w is essential. Indeed, let us consider the situation where there exists a geodesic γ, connecting x^- to x^+, such that $W(\gamma)$ is always positive except at the end-points $x^\pm$ and at a third point $z \in X$ where $W(z) = 0$. If z is singular, in the sense that $\liminf\limits_{y \to z} \frac{w(y)}{|y|} > 0$ (in particular $w = \sqrt{W}$ is not Lipschitz), then there exists a heteroclinic connexion $\gamma^- : x^- \to z$ which reaches z in finite time (up to translation, one can assume that $\gamma^-(t) = z$ for $t \geq 0$) since the corresponding solution of the ODE (A.3.4) reaches its maximum in finite time. Similarly, there exists a heteroclinic connexion $\gamma^+ : z \to x^+$ such that $\gamma^+(t) = z$ for $t \leq 0$. Thus, there exists a heteroclinic connexion between x^- and x^+ obtained by matching γ^- and γ^+.

In fact, the assumption on the geodesic connecting x^- to x^+ in Proposition A.3.2 could be weakened as follows. Assume that there exists a geodesic $\gamma : x^- \to x^+$ which meets S at n points $z_1 = x^-$, z_2,…,$z_n = x^+$. If $\liminf\limits_{y \to z_i} \frac{w(y)}{|y|} > 0$ for $i = 2, \dots, n-1$, then there exists a heteroclinic connexion between x^- and x^+.

Proof. The key of the proof is to study the ODE (A.3.4) for which solutions, with initial conditions, are unique if w is Lipschitz. Assume by contradiction that γ is a global minimizer of E_W under the constraint $\gamma : x^- \to x^+$. Then, both Proposition A.3.2 and Lemma A.3.5 imply that $|\dot\gamma|^2(t) = W(\gamma(t))$ a.e. and $E_W(\gamma) = c_W(x^-, x^+) = d_w(x^-, x^+) = L_w(\gamma)$. In particular, γ minimizes L_w, i.e. γ is a geodesic in (X, d_w). Let φ be the parametrization of γ by arc length in (X, d): $\varphi(t) := \pm L(\gamma_{|(0,t)})$ whenever $\pm t \geq 0$. $\varphi : \mathbb{R} \to \mathbb{R}$ is non decreasing, $\mathcal{C}^1$ and one has

$$\varphi'(t) = |\dot\gamma|(t) = w(\gamma(t)) \text{ a.e.}$$

Let $\gamma_0 : I \to X$ be the parameterization of γ by arc length: $I := \varphi(\mathbb{R})$ and $\gamma_0(\varphi(t)) := \gamma(t)$. Then, since $\varphi' = w \circ \gamma$, φ is a solution of the Cauchy problem:

$$\begin{cases} \varphi'(t) = (w \circ \gamma_0)(\varphi(t)). \\ \varphi(t_0) = \varphi_0, \end{cases} \tag{A.3.6}$$

with $t_0 = \varphi_0 = 0$. Note that $\gamma_0 : \varphi(\mathbb{R}) \to (X, d)$ is 1-Lipschitz as it is parameterized by arc length. Moreover $w : (X, d) \to \mathbb{R}^+$ is locally Lipschitz by assumption. Thus $w \circ \gamma_0$ is locally Lipschitz on $\mathbb{R}$ and solutions of (A.3.6) are unique. However, since γ is a geodesic in (X, d_w) connecting x^- to x^+, by assumption, there exists $t_0 \in \mathbb{R}$ such that $0 = w(\gamma(t_0)) = (w \circ \gamma_0)(\varphi(t_0))$ and $\gamma(t_0) \neq x^\pm$. By uniqueness, one has $\varphi \equiv \varphi(t_0)$. In particular, $L_w(\gamma) = \varphi(+\infty) - \varphi(-\infty) = 0$ and $x^- = x^+$ which is a contradiction. ☐

Remark A.3.7. We have proved that every global minimizer of E_W under the constraint $\gamma : x^- \to x^+$ is obtained by reparametrization of a geodesic in (X, d_w). Moreover, the parameterization φ must satisfy the ODE (A.3.6).

Corollary A.3.8. *Fix* $d \geq 1$ *the dimension and let* $W : \mathbb{R}^d \to \mathbb{R}^+$ *be a potential. Let us fix two wells* $u^\pm = (a, \overline{u}^\pm) \in \mathbb{R}^d$*. Assume that* $W_a(\cdot) := W(a, \cdot)$*, defined on* $\mathbb{R}^d_a := \{(a, u) \ : \ u \in \mathbb{R}^{d-1}\}$*, satisfies the following conditions:*

1. $d_* := \inf\{|u - v| \ : \ u, v \in \mathbb{R}^d_a,\ W(u) = W(v) = 0$ *and* $u \neq v\} > 0$.
2. $\omega(\varepsilon) := \inf\{W_a(u) \ : \ u \in \mathbb{R}^d_a,\ d(u, S_a) > \varepsilon\} > 0$ *for all* $\varepsilon > 0$*, where* $S_a = \{u \in \mathbb{R}^d_a \ : \ W(u) = 0\}$.
3. *There exists a geodesic* $\gamma_0 : \overline{u}^- \to \overline{u}^+$ *in the space* $(\mathbb{R}^d_a, d_w)$ *such that* $\mathrm{Im}(\gamma_0) \cap S_a = \{\overline{u}^\pm\}$ *where* $w := \sqrt{W_a}$.

Then the following one-dimensional minimization problem has a solution:

$$\inf \left\{ E^{1D}(\gamma) := \frac{1}{2} \int_{\mathbb{R}} |\gamma'(t)|^2 + W(a, \gamma(t))\, \mathrm{d}t \ : \right. \\ \left. \gamma \in H^1_{loc}(\mathbb{R}, \mathbb{R}^{d-1}),\ \gamma(\pm\infty) = \overline{u}^\pm \right\}. \tag{A.3.7}$$

Remark A.3.9. In dimension $d = 2$, $\mathbb{R}^d_a$ is a line and the third assumption on the potential is easily understandable. Indeed, for any curve $\gamma_0 : \overline{u}^+ \to \overline{v}^-$, $\mathrm{Im}(\gamma_0)$ is a segment containing $[\overline{u}^+, \overline{v}^-]$. Moreover if γ_0 is optimal, $\mathrm{Im}(\gamma_0) = [\overline{u}^+, \overline{v}^-]$. Thus, third assumption means that $W(u) > 0$ on the open interval $(\overline{u}^+, \overline{v}^-)$.

Proof. Note that Corollary A.3.8 is not a strict application of Proposition A.3.2 since the energy E_W defined in (A.3.1) is defined on locally Lipschitz maps while E^{1D} above is defined over H^1 curves. However, note that if γ is the global minimizer given by the second claim in Proposition A.3.2, then γ satisfies $E_{W_a}(\gamma) = L_w(\gamma)$ and γ minimizes the length:

$$L_w(\gamma) = \inf\{L_w(\sigma) \ : \ \sigma \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}^d_a) \text{ s.t. } \sigma : \overline{u}^- \to \overline{u}^+\}.$$

Thus, as the inequality $L_w(\sigma) \leq E_{W_a}(\sigma)$ is obviously true for $\sigma \in H^1_{loc}(\mathbb{R}, \mathbb{R}^d_a)$, γ is also a minimizer for (A.3.7). $\square$

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Résumé

Cette thèse est consacrée à l'étude de certains problèmes variationnels de type transition de phase vectorielle ou "phase-field" qui font intervenir une contrainte de divergence. Ces modèles sont généralement basés sur une énergie dépendant d'un paramètre qui peut représenter une grandeur physique négligeable ou qui est liée à une méthode d'approximation numérique par exemple. Une question centrale concerne alors le comportement asymptotique de ces énergies et des minimiseurs globaux ou locaux lorsque ce paramètre tend vers 0. Cette thèse présente différentes stratégies prenant en compte la contrainte de divergence. Elles seront illustrées à travers l'étude de deux modèles. Le premier est une approximation du modèle Eulérien pour le transport branché par un modèle de type phase-field avec divergence prescrite. Nous montrons comment une estimation uniforme de l'énergie, en fonction de la contrainte sur la divergence, permet d'établir un résultat de Γ-convergence. Le second modèle, en lien avec la théorie du micromagnétisme, concerne des énergies de type Aviles-Giga dans un cadre vectoriel avec contrainte de divergence. Nous illustrerons dans quelle mesure la méthode d'entropie permet de caractériser les minimiseurs globaux. Dans certaines situations nous montrerons une conjecture de type De Giorgi concernant la symétrie $1D$ des minimiseurs globaux de l'énergie sous contrainte au bord.

Mots clés : Calcul des Variations, Γ-convergence, Problèmes à discontinuité libre, Transition de phase, Ginzburg-Landau, Transport branché

Abstract

This thesis is devoted to the study of phase-field type variational models with divergence constraint. These models typically involve an energy depending on a parameter which represents a negligible physical quantity or is linked to some numerical approximation method for instance. A central question concerns the asymptotic behavior of these energies and of their global or local minimizers when this parameter goes to 0. We present different strategies which allow to take the divergence constraint into account. They will be illustrated in two models. The first one is a phase-field type approximation, involving a divergence constraint, of the Eulerian model for branched transportation. We illustrate how uniform estimates on the energy, depending on the constraint on the divergence, allow to establish a Γ-convergence result. The second model, related to micromagnetics, concerns Aviles-Giga type energies for divergence-free vector fields. We use the entropy method in order to characterize global minimizers. In some situations, we will prove a De Giorgi type conjecture concerning the one-dimensional symmetry of global minimizers under boundary condition.

Keywords: Calculus of Variations, Γ-convergence, Free discontinuity problems, Phase transition, Ginzburg-Landau, Branched transportation