2 Truth Tables, Equivalences and the Contrapositive

2.1 Truth Tables

In a mathematical system, true and false statements are the statements of the system, and the label 'true' or 'false' associated with a given statement is its *truth value.*

Notation.

- *We use the symbol* $\neg$ *to mean* ***not****.*
- *We use the symbol* $\wedge$ *to mean* ***and****.*
- *We use the symbol* $\vee$ *to mean* ***or****.*
- *We use the symbol* $\implies$ *to mean* ***implies****.*
- *We use the symbol* $\iff$ *to mean* ***if and only if****.*

Remark. *Recall that for two statements* P *and* Q*,* $P \iff Q$ *means that*

$$P \implies Q \textbf{ and } Q \implies P.$$

When $P \iff Q$*, we say* P *and* Q *are* ***equivalent****.*

Example. *Let* $x \in \mathbb{Z}$*, let* P *and* Q *be the following statements:*

$$P : x \geq 5 \qquad Q : x \leq 7$$

Then $P \wedge Q$ *represents*

$$\begin{aligned} P \wedge Q &: x \geq 5 \text{ and } x \leq 7 \\ &: 5 \leq x \leq 7 \end{aligned}$$

Example. *Let* P *and* Q *be two statements.* $P \implies Q$ *is the statement that 'P implies Q', or equivalently it is the statement that 'If P, then Q'. In a truth table, we consider all possible combinations of the truth values of* P *and* Q *and the consequent truth value of the statement* $P \implies Q$*:*

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

Remark. *Here it is important to note that* $P \implies Q$ *does* ***not*** *mean that* P *causes* Q*. It merely says that* P *is a sufficient condition for* Q*.*

Example. *Let* P *and* Q *be two statements. We have that* $P \wedge Q$ *is true exactly when* P *and* Q *are both true. The corresponding truth table is as follows.*

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Example. *Let P and Q be two statements. We have that $P \vee Q$ is true exactly when P or Q is true. Hence, we have the following corresponding truth table.*

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Example. *Let P and Q be two statements. The corresponding truth table for $(\neg P) \vee Q$ is as follows.*

P	$\neg P$	Q	$(\neg P) \vee Q$
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T

Example. *Let P and Q be two statements. The corresponding truth table for $(\neg Q) \implies (\neg P)$ is as follows.*

P	Q	$\neg P$	$\neg Q$	$(\neg Q) \implies (\neg P)$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

2.2 Logical Equivalence and the Contrapositive

If we take two statements which are logically equivalent, say P is equivalent to Q, then proving P to be true is equivalent to proving Q to be true. Similarly, proving Q to be true is equivalent to proving P to be true.

We have the following theorem.

Theorem 2.1. *Let P and Q be two statements.*

(i) $P \implies Q$ is equivalent to $(\neg Q) \implies (\neg P)$.

(ii) $P \implies Q$ is equivalent to $(\neg P) \vee Q$.

Proof. We prove part (i) and leave part (ii) as Exercise 2.1 a). We have the following truth table.

P	$\neg P$	Q	$\neg Q$	P $\implies$ Q	$(\neg Q) \implies (\neg P)$
T	F	T	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	T	F	T	T	T

Hence, for all truth values of P and Q, we have that $P \implies Q$ and $(\neg Q) \implies (\neg P)$ have the same truth values. Therefore, $P \implies Q$ is equivalent to $(\neg Q) \implies (\neg P)$. □

We have the following definition.

Definition 2.2. *Let P and Q be two statements. The* ***contrapositive*** *of the statement $P \implies Q$ is given by the statement $(\neg Q) \implies (\neg P)$.*

Remark. *As seen in Theorem 2.1, we have that the statement $P \implies Q$ is equivalent to its contrapositive.*

The proof of the following theorem demonstrates an example of a proof by contrapositive.

Theorem 2.3 (Pigeonhole Principle)**.** *Let A be a set with n elements and B a set with m elements where $m, n \in \mathbb{N}$ with $m < n$. Then there is no injection from A into B.*

We will give two proofs of Theorem 2.3. The first one is a proof by contradiction, and the second one is a proof by contrapositive.

Proof 1. Let $A = \{a_1, a_2, \dots, a_n\}$ and $B = \{b_1, b_2, \dots, b_m\}$ with $m < n$. Let $g : A \to B$. To the contrary, suppose that $g : A \to B$ is injective. Now, define a set

$$C = \{g(a_i) : 1 \leq i \leq n, n \in \mathbb{N}\}.$$

Then C is a subset of B, and since g is injective, C is a set with n elements. But then C is a subset of B which contains more elements than B, which is a contradiction. Therefore, there is no injection from A to B. □

Proof 2. We will prove that if there is an injection from A into B, then $m \geq n$.

Let $A = \{a_1, a_2, \dots, a_n\}$ and $B = \{b_1, b_2, \dots, b_m\}$ and suppose that $g : A \to B$ is injective. Now, define

$$C = \{g(a_i) : 1 \leq i \leq n, n \in \mathbb{N}\}.$$

Then C is a subset of B, and since g is injective, C is a set with n elements. It follows that B must have at least n elements, that is $m \geq n$. The result follows by contrapositive. □

Definition 2.4. *Let P and Q be two statements. The* ***converse*** *of the statement $P \implies Q$ is given by $Q \implies P$.*

Remark. *We have that $P \implies Q$ is* ***not*** *equivalent to its converse.*

We have the following proposition.

Proposition 2.5. *Suppose P is a statement. Then $P \iff (\neg(\neg P))$.*

Proof. We have the following truth table.

P	$\neg P$	$\neg(\neg P)$
T	F	T
F	T	F

We can see from the table that the truth values of P and $\neg(\neg P)$ always agree. Hence, the two statements P and $\neg(\neg P)$ are equivalent. □

The next proposition shows that $\wedge$ and $\vee$ are *associative*, that is, $P \wedge Q \wedge R$ and $P \vee Q \vee R$ are statements that do not require parentheses.

Proposition 2.6. *Suppose P, Q, R are statements.*

(i) $((P \wedge Q) \wedge R) \iff (P \wedge (Q \wedge R))$.

(ii) $((P \vee Q) \vee R) \iff (P \vee (Q \vee R))$.

Proof. We prove part (i) and leave the proof of part (ii) as Exercise 2.1 b). We have the following truth table.

P	Q	R	P ∧ Q	$(P \wedge Q) \wedge R$	$Q \wedge R$	$P \wedge (Q \wedge R)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	F	F	T	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

Hence, for any truth values of P, Q, R, the truth table shows that $((P \wedge Q) \wedge R) \iff (P \wedge (Q \wedge R))$. □

The above proposition shows that we can write $P \wedge Q \wedge R$ and $P \vee Q \vee R$ without parentheses and without there being confusion. As an exercise, one may also prove the following sometimes useful equivalences.

Proposition 2.7. *Let P, Q, R be statements. Then*

(i) $(P \wedge (Q \wedge R)) \iff ((P \wedge Q) \wedge (P \wedge R))$.

(ii) $(P \vee (Q \vee R)) \iff ((P \vee Q) \vee (P \vee R))$.

Proof. Exercise 2.3. □

Proposition 2.8. *Let P, Q, R be statements. Then*

(i) $(P \wedge (Q \vee R)) \iff ((P \wedge Q) \vee (P \wedge R))$.

(ii) $(P \vee (Q \wedge R)) \iff ((P \vee Q) \wedge (P \vee R))$.

Proof. We will prove part (i) and leave the proof of part (ii) as Exercise 2.1 c). We have the following truth table.

P	Q	R	$Q \vee R$	$P \wedge (Q \vee R)$	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Hence, for any truth values of P, Q, R, the above truth table shows that $(P \wedge (Q \vee R)) \iff ((P \wedge Q) \vee (P \wedge R))$. □

Theorem 2.9. *Suppose P, Q are two statments. Then* *(i)* $\neg(P \wedge Q) \iff ((\neg P) \vee (\neg Q))$.

(ii) $\neg(P \vee Q) \iff ((\neg P) \wedge (\neg Q))$.

(iii) $\neg(P \implies Q) \iff (P \wedge (\neg Q))$.

Proof. We will prove part (i) and leave part (ii) and (iii) as Exercise 2.2. We have the following truth table.

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$\neg(P \wedge Q)$	$(\neg P) \vee (\neg Q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Thus, for any truth values of P, Q, R, the truth values of $\neg(P \wedge Q)$ and $(\neg P) \vee (\neg Q)$ are the same. □

We have the following proposition.

Proposition 2.10. *Let P, Q be two statements. Then*

$$(P \vee Q) \iff ((\neg P) \implies Q)$$

Proof. Exercise 2.4 a). □

Remark. *Let P, Q, R be statements. Then*

$$P \implies Q \iff R \qquad \text{and} \qquad P \iff Q \implies R$$

have no clear meanings. In Exercise 2.5, one shows that the statements

$$P \implies (Q \iff R) \qquad \text{and} \qquad (P \implies Q) \iff R$$

*are **not** equivalent and that the statements*

$$P \iff (Q \implies R) \qquad \text{and} \qquad (P \iff Q) \implies R$$

*are **not** equivalent. Hence, the meaning of assertions such as*

$$P \implies Q \iff R \implies S$$

is undefined.