3 Negations and Contrapositives of Statements with Quantifiers

3.1 Negations

Let X be a set, and suppose that $P(x)$ is a statement involving $x \in X$. Suppose that

$$\forall\, x \in X, P(x)$$

is a **false** statement. Then there must be at least one $x \in X$ such that $P(x)$ does not hold. That is,

$$\neg(\forall\, x \in X,\ P(x)) \quad \Longrightarrow \quad \exists\, x \in X \text{ such that } \neg P(x) \tag{1}$$

Conversely, suppose that

$$\exists\, x \in X \text{ such that } \neg P(x)$$

is a true statement. Then it is not the case that $P(x)$ holds for all $x \in X$, that is

$$\exists\, x \in X \text{ such that } \neg P(x) \quad \Longrightarrow \quad \neg(\forall\, x \in X,\ P(x)) \tag{2}$$

By (1) and (2), it follows that

$$\neg(\forall\, x \in X,\ P(x) \quad \Longleftrightarrow \quad \exists\, x \in X \text{ such that } \neg P(x)$$

Equivalently, we have

$$\begin{aligned} \neg(\neg(\forall\, x \in X,\ P(x))) \quad &\Longleftrightarrow \quad \neg(\exists\, x \in X \text{ such that } \neg P(x)) \\ &\Longleftrightarrow \quad \forall\, x \in X,\ P(x) \end{aligned}$$

Setting $Q(x) = \neg P(x)$, we have

$$\neg(\exists\, x \in X \text{ such that } Q(x)) \quad \Longleftrightarrow \quad \forall\, x \in X,\ \neg Q(x)$$

Example. *Let X be a non-empty set, and for $x \in X$, let $P(x)$ be the following statement.*

$$\forall\, x' \in X,\ x \neq x' \quad \Longrightarrow \quad f(x) \neq f(x')$$

Recall that a function $f : X \to Y$ is injective if and only if

$$\forall\, x \in X, P(x)$$

We know that

$$\neg(\forall\, x \in X,\ P(x)) \quad \Longleftrightarrow \quad \exists\, x \in X \textit{ such that } \neg P(x).$$

Now let $Q(x, x')$ be the following statement.

$$x \neq x' \quad \Longrightarrow \quad f(x) \neq f(x')$$

Then

$$\begin{aligned} \neg P(x) \quad &\Longleftrightarrow \quad \neg(\forall\, x' \in X,\ Q(x, x')) \\ &\Longleftrightarrow \quad \exists\, x' \in X \textit{ such that } \neg Q(x, x') \end{aligned}$$

Further, using results from Section 2, we have that

$$\begin{aligned} \neg Q(x,x') &\iff (x \neq x') \wedge (\neg(f(x) \neq f(x'))) \\ &\iff (x \neq x') \wedge (f(x) = f(x')) \end{aligned}$$

Summarising, we have that $f : X \to Y$ *is* **not** *injective if and only if*

$$\exists\ x, x' \in X \text{ such that } (x \neq x') \wedge (f(x) = f(x')).$$

Example. *Suppose that* $f : X \to Y$. *By definition, we know that* f *is surjective if and only if*

$$\forall\ y \in Y,\ \exists\ x \in X \text{ such that } f(x) = y$$

Let $P(y)$ *be the following statement.*

$$\exists\ x \in X \text{ such that } f(x) = y$$

Then

$$\begin{aligned} f \text{ is not surjective} &\iff \neg(\forall\ y \in Y,\ P(y) \\ &\iff \exists\ y \in Y \text{ such that } \neg P(y) \\ &\iff \exists\ y \in Y \text{ such that } \neg(\exists\ x \in X \text{ such that } f(x) = y) \\ &\iff \exists\ y \in Y \text{ such that } \forall\ x \in X,\ f(x) \neq y \end{aligned}$$

Example. *Let* $a_n \in \mathbb{R}$, *for every* $n \in \mathbb{N}$. *Consider the following statement.*

$$\exists\ c \in \mathbb{R} \text{ such that } \forall\ \varepsilon > 0\ \exists\ N \in \mathbb{N} \text{ such that } \forall\ n \geq N\ |a_n - c| < \varepsilon$$

We negate this statement in a series of steps such that each consecutive pair of statements is clearly equivalent.

$$\begin{aligned} &\neg(\exists\ c \in \mathbb{R} \text{ such that } \forall\ \varepsilon > 0\ \exists\ N \in \mathbb{N} \text{ such that } \forall\ n \geq N\ |a_n - c| < \varepsilon) \\ &\iff \forall\ c \in \mathbb{R}\ \neg(\forall\ \varepsilon > 0\ \exists\ N \in \mathbb{N} \text{ such that } \forall\ n \geq N\ |a_n - c| < \varepsilon) \\ &\iff \forall\ c \in \mathbb{R}\ \exists\ \varepsilon > 0 \text{ such that } \neg(\exists\ N \in \mathbb{N} \text{ such that } \forall\ n \geq N\ |a_n - c| < \varepsilon) \\ &\iff \forall\ c \in \mathbb{R}\ \exists\ \varepsilon > 0 \text{ such that } \forall\ N \in \mathbb{N}\ \neg(\forall\ n \geq N\ |a_n - c| < \varepsilon) \\ &\iff \forall\ c \in \mathbb{R}\ \exists\ \varepsilon > 0 \text{ such that } \forall\ N \in \mathbb{N}\ \exists\ n \geq N \text{ such that } \neg(|a_n - c| < \varepsilon) \\ &\iff \forall\ c \in \mathbb{R}\ \exists\ \varepsilon > 0 \text{ such that } \forall\ N \in \mathbb{N}\ \exists\ n \geq N \text{ such that } |a_n - c| \geq \varepsilon. \end{aligned}$$

Example. *Let* $[0,1) = \{x \in \mathbb{R} :\ 0 \leq x < 1\ \}$ *and* $(0,1) = \{x \in \mathbb{R} :\ 0 < x < 1\ \}$. *Define* $f : [0,1) \to (0,1)$ *by*

$$f(x) = \begin{cases} 1 - \frac{1}{n+1}, & \text{if } x = 1 - \frac{1}{n} \text{ for some } n \in \mathbb{N}; \\ x, & \text{otherwise.} \end{cases}$$

We have the following meaning of 'otherwise'.

$$\begin{aligned} \neg\left(x = 1 - \tfrac{1}{n} \text{ for some } n \in \mathbb{N}\right) &\iff \neg\left(\exists\ n \in \mathbb{N} \text{ such that } x = 1 - \tfrac{1}{n}\right) \\ &\iff \forall\ n \in \mathbb{N}\ \neg\left(x = 1 - \tfrac{1}{n}\right) \\ &\iff \forall\ n \in \mathbb{N}\ x \neq 1 - \tfrac{1}{n} \end{aligned}$$

3.2 Contrapositives of statements with quantifiers

Let X be a set. Suppose that $P(x)$ and $Q(x)$ are two statements involving $x \in X$. We have seen that $P(x) \implies Q(x)$ is equivalent to its contrapositive: $\neg Q(x) \implies \neg P(x)$. Hence

$$(\forall\, x \in X, P(x) \implies Q(x)) \qquad \iff \qquad (\forall\, x \in X,\ \neg Q(x) \implies \neg P(x))$$

Similarly,

$$(\exists\, x \in X \text{ such that } P(x) \implies Q(x)) \qquad \iff \qquad (\exists\, x \in X \text{ such that } \neg Q(x) \implies \neg P(x))$$

We have the following theorem.

Theorem 3.1. *Let X, Y be sets. Suppose $f : X \to Y$. The map f is injective if and only if*

$$\forall\, x, x' \in X,\ f(x) = f(x') \implies x = x'.$$

Proof. By definition,

$$f \text{ is injective} \qquad \iff \qquad \left(\forall\, x, x' \in X,\ x \neq x' \implies f(x) \neq f(x')\right).$$

The contrapositive of the statement

$$x \neq x' \implies f(x) \neq f(x')$$

is given by

$$\neg\left(f(x) \neq f(x')\right) \implies \neg\left(x \neq x'\right),$$

or equivalently,

$$f(x) = f(x') \implies x = x'.$$

It follows that

$$\forall\, x, x' \in X,\ x \neq x' \implies f(x) \neq f(x')$$

is equivalent to

$$\forall\, x, x \in X,\ f(x) = f(x') \implies x = x'.$$

The result follows. □