4 Set Operations

Suppose that A and B are subsets of some set X.

Notation.

- $A \cup B$ *denotes the* ***union*** *of* A *and* B*, that is*

$$A \cup B = \{x \in X : \ x \in A \text{ or } x \in B \ \}.$$

- $A \cap B$ *denotes the* ***intersection*** *of* A *and* B*, that is*

$$A \cap B = \{x \in X : \ x \in A \text{ and } x \in B \ \}.$$

  When $A \cap B = \emptyset$*, we say that* A *and* B *are* ***disjoint****.*

- $A \setminus B$ *denotes the* ***relative complement*** *of* A *with respect to* B*, that is*

$$A \setminus B = \{x \in X : \ x \in A \text{ and } x \notin B \ \}.$$

- A^c *denotes the* ***complement*** *of* A*, that is*

$$A^c = \{x \in X : \ x \notin A \ \}.$$

We have the following theorem.

Theorem 4.1. *Let* X *be a set, and for* $x \in X$*, let* $P(x)$ *be the statement that* x *satisfies the criteria* P*, and let* $Q(x)$ *be the statement that* x *satisfies the criteria* Q*. Set*

$$A = \{x \in X : P(x)\} \qquad \text{and} \qquad B = \{x \in X : Q(x)\}.$$

Then

$$\begin{aligned} A \cap B &= \{x \in X : P(x) \wedge Q(x)\}, \\ A \cup B &= \{x \in X : P(x) \vee Q(x)\}. \end{aligned}$$

Proof. For $x \in X$, we have that $x \in A$ if and only if the statement $P(x)$ holds. Similarly, we have that $x \in B$ if and only if the statement $Q(x)$ holds. Then

$$\begin{aligned} A \cap B &= \{x \in X : (x \in A) \wedge (x \in B)\} \\ &= \{x \in X : P(x) \wedge Q(x)\} \end{aligned}$$

and

$$\begin{aligned} A \cup B &= \{x \in X : (x \in A) \vee (x \in B)\} \\ &= \{x \in X : P(x) \vee Q(x)\}. \end{aligned}$$

□

We have the following proposition

Proposition 4.2. *Suppose* A, B, C *are subsets of a set* X*. Then*

(i) $A \cap (B \cap C) = (A \cap B) \cap C$*.*

(ii) $A \cup (B \cup C) = (A \cup B) \cup C$.

Proof. We will prove part (i) and leave part (ii) as Exercise 3.2 a).

Suppose $x \in X$. Let P be the statement that $x \in A$, let Q be the statement that $x \in B$, and let R be the statement that $x \in C$. Recall that $P \wedge (Q \wedge R) \iff (P \wedge Q) \wedge R$. Then

$$\begin{aligned} x \in A \cap (B \cap C) &\iff (x \in A) \wedge (x \in B \cap C) \\ &\iff (x \in A) \wedge ((x \in B) \wedge (x \in C)) \\ &\iff P \wedge (Q \wedge R) \\ &\iff (P \wedge Q) \wedge R \\ &\iff ((x \in A) \wedge (x \in B)) \wedge (x \in C) \\ &\iff (x \in A \cap B) \wedge (x \in C) \\ &\iff x \in (A \cap B) \cap C. \end{aligned}$$

Hence, we have that $x \in A \cap (B \cap C)$ if and only if $x \in (A \cap B) \cap C$. It follows that $A \cap (B \cap C) = (A \cap B) \cap C$. □

We have the following theorem.

Theorem 4.3. *Let A, B, C be subsets of a set X. Then*

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof. We will prove part (i) and leave part (ii) as Exercise 3.2 b).

Suppose $x \in X$. Let P be the statement that $x \in A$, let Q be the statement that $x \in B$, and let R be the statement that $x \in C$. Recall that $P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R)$. Then

$$\begin{aligned} x \in A \cap (B \cup C) &\iff (x \in A) \wedge (x \in B \cup C) \\ &\iff (x \in A) \wedge ((x \in B) \vee (x \in C)) \\ &\iff P \wedge (Q \vee R) \\ &\iff (P \wedge Q) \vee (P \wedge R) \\ &\iff ((x \in A) \wedge (x \in B)) \vee ((x \in A) \wedge (x \in C)) \\ &\iff (x \in A \cap B) \vee (x \in A \cap C) \\ &\iff x \in (A \cap B) \cup (A \cap C). \end{aligned}$$

Hence, we have that $x \in A \cap (B \cup C)$ if and only if $x \in (A \cap B) \cup (A \cap C)$. It follows that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. □

We have the following proposition.

Proposition 4.4. *Suppose A, B are subsets of a set X. Then*

(i) $A \setminus B = A \cap B^c$.

(ii) $(A \setminus B)^c = A^c \cup B$.

Proof. We will prove part (i) and leave part (ii) as Exercise 3.3 c).
Let $x \in X$. Then

$$x \in A \setminus B \iff (x \in A) \wedge (x \notin B) \iff (x \in A) \wedge (x \in B^c) \iff x \in A \cap B^c,$$

Hence, we have that $x \in A \setminus B$ is equivalent to $x \in A \cap B^c$. It follows that $A \setminus B = A \cap B^c$. □

We have the following theorem.

Theorem 4.5 (De Morgan's Laws). *Suppose that A, B, C are subsets of a set X. Then*

(i) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

(ii) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

(iii) $(A \cap B)^c = A^c \cup B^c$.

(iv) $(A \cup B)^c = A^c \cap B^c$.

Proof. We will prove parts (i) and (iv) and leave parts (ii) as Exercise 3.3 d) and part (iii) as Exercise 3.3 e).
(i) Let $x \in X$. Recall that for statements P, Q, R, we have that $P \wedge (Q \wedge R) \iff (P \wedge Q) \wedge (P \wedge R)$. Then

$$\begin{aligned} x \in A \setminus (B \cup C) &\iff (x \in A) \wedge (x \notin B \cup C) \\ &\iff (x \in A) \wedge (\neg(x \in B \cup C)) \\ &\iff (x \in A) \wedge (\neg((x \in B) \vee (x \in C))) \\ &\iff (x \in A) \wedge ((x \notin B) \wedge (x \notin C)) \\ &\iff ((x \in A) \wedge (x \notin B)) \wedge ((x \in A) \wedge (x \notin C)) \\ &\iff (x \in A \setminus B) \wedge (x \in A \setminus C) \\ &\iff x \in (A \setminus B) \cap (A \setminus C). \end{aligned}$$

Hence, we have that $x \in A \setminus (B \cup C)$ if and only if $x \in (A \setminus B) \cap (A \setminus C)$. It follows that $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

(iv) Let $x \in X$. Then

$$\begin{aligned} x \in (A \cup B)^c &\iff \neg(x \in A \cup B) \\ &\iff \neg((x \in A) \vee (x \in B)) \\ &\iff \neg((x \in A) \wedge (\neg(x \in B))) \\ &\iff (x \in A^c) \wedge (x \in B^c) \\ &\iff x \in A^c \cap B^c. \end{aligned}$$

Hence, we have that $x \in (A \cup B)^c$ if and only if $x \in A^c \cap B^c$. It follows that $(A \cup B)^c = A^c \cap B^c$. □

Notation. *It is often convenient to denote the elements of a set using indices. For example, suppose A is a set with 5 elements. Then we can denote these elements as a_1, a_2, a_3, a_4, a_5. So we can write*

$$A = \{a_i : \ i \in I\}, \text{ where } I = \{1, 2, 3, 4, 5\}.$$

*The set I is called the **indexing set**.*

Example. *There are infinitely many primes within the set of integers. Ordering the primes in increasing order and letting p_i denote the ith prime, for $i \in \mathbb{N}$, we have that the set of all primes is given by*

$$\{p_i : i \in \mathbb{N}\}.$$

Alternatively, we sometimes denote this set by $\{p_i\}_{i\in\mathbb{N}}$.

Notation. *Let $\{A_i\}_{i\in I}$ be a collection of subsets of a set X where I is an indexing set. Then we write $\bigcup_{i\in I} A_i$ to denote the* ***union of all the sets*** *A_i, for $i \in I$. That is,*

$$\bigcup_{i\in I} A_i = \{x \in X : \exists\ i \in I \text{ such that } x \in A_i\}.$$

Further, we write $\bigcap_{i\in I} A_i$ to denote the ***intersection of all the sets*** *A_i, for $i \in I$. That is,*

$$\bigcap_{i\in I} A_i = \{x \in X : \forall\ i \in I,\ x \in A_i\}.$$

We have the following proposition.

Proposition 4.6. *Let X be a set, let A be a subset of X, and let $\{B_i\}_{i\in I}$ be an indexed collection of subsets, where I is an indexing set. Then we have*

(i) $A \setminus \bigcap_{i\in I} B_i = \bigcup_{i\in I} (A \setminus B_i)$.

(ii) $A \setminus \bigcup_{i\in I} B_i = \bigcap_{i\in I} (A \setminus B_i)$.

Proof. We will prove part (i) and leave part (ii) as Exercise 3.4.
We know that $x \in \bigcap_{i\in I} B_i$ if and only if we have that $x \in B_i$, for all $i \in I$. Then $x \notin \bigcap_{i\in I} B_i$ if and only if there exists an $i \in I$ such that $x \notin B_i$. Now, suppose that $x \in A \setminus \bigcap_{i\in I} B_i$. Then $x \in A$, and for some $i \in I$, we have that $x \notin B_i$. Hence, for some $i \in I$, we have that $x \in A \setminus B_i$. Then $x \in \bigcup_{i\in I}(A \setminus B_i)$ which shows that $A \setminus \bigcap_{i\in I} B_i \subseteq \bigcup_{i\in I}(A \setminus B_i)$.

Now, suppose that $x \in \bigcup_{i\in I}(A\setminus B_i)$. Hence, for some $i \in I$, we have that $x \in A\setminus B_i$. Then for some $i \in I$, we have that $x \in A$ and $x \notin B_i$. Since there exists some $i \in I$ such that $x \notin B_i$, we have $x \notin \bigcap_{i\in I} B_i$. Then $x \in A \setminus \bigcap_{i\in I} B_i$ which shows that $\bigcup_{i\in I}(A \setminus B_i) \subseteq A \setminus \bigcap_{i\in I} B_i$. Summarising the above, we have that $A \setminus \bigcap_{i\in I} B_i = \bigcup_{i\in I}(A \setminus B_i)$. □

We have the following proposition.

Proposition 4.7. *Suppose that $f : X \to Y$ is bijective, and let $A \subseteq X$ and $B = X \setminus A$. Then $f[A] \cap f[B] = \emptyset$.*

Proof. To the contrary, suppose that there exists some $y \in Y$ such that $y \in f[A] \cap f[B]$. Then $y \in f[A]$, so there exists some $a \in A$ such that $y = f(a)$. Similarly, $y \in f[B]$, so there exists some $b \in B$ such that $y = f(b)$. Hence $f(a) = y = f(b)$.

Since f is bijective, we know that f^{-1} exists [so $f^{-1} \circ f$ is the identity map on X]. Hence, we have that

$$f^{-1}(y) = f^{-1}(f(a)) = (f^{-1} \circ f)(a) = a,$$

for all $y \in Y$. Similarly,

$$f^{-1}(y) = f^{-1}(f(b)) = (f^{-1} \circ f)(b) = b,$$

for all $y \in Y$. This implies that $a = b$, which means b is an element of A (since $a \in A$). But $b \in B$, which means that $b \notin A$ by the definition of B, which is a contradiction since we cannot have both $b \in A$ and $b \notin A$. Hence, it is impossible to have $y \in f[A] \cap f[B]$, which means $f[A] \cap f[B]$ must be the empty set. □

We have the following theorem.

Theorem 4.8. *Suppose that $f : X \to Y$ and $X = U \cup V$. Then*

(i) $f[X] = f[U] \cup f[V]$.

(ii) if f is injective and $U \cap V = \emptyset$, then $f[U] \cap f[V] = \emptyset$.

Proof.

(i) Since $U, V \subseteq X$, we have that $f[U], f[V] \subseteq f[X]$, so $f[U] \cup f[V] \subseteq f[X]$. On the other hand, take $x \in X$. Then $x \in U$ or $x \in V$, so $f(x) \in f[U]$ or $f(x) \in f[V]$. Therefore $f(x) \in f[U] \cup f[V]$. Since this holds for all $x \in X$, we have $f[X] \subseteq f[U] \cup f[V]$. Hence, $f[X] = f[U] \cup f[V]$.

(ii) Exercise 3.5.

□

We have the following definition.

Definition 4.9. *Suppose that $f : X \to Y$ and $V \subseteq Y$. We define the* ***inverse image*** *of V under f by*

$$f^{-1}[V] = \{x \in X : \ f(x) \in V\}.$$

Remark. *The above notation does* **not** *mean that f has an inverse or that f^{-1} is necessarily a function!*

Remark. *We have that $f^{-1}[\emptyset] = \emptyset$.*

Example. *Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $f((x, y)) = 2x - 5y$. Then the* ***kernel*** *of f is given by*

$$\begin{aligned} f^{-1}[\{0\}] &= \{(x, y) \in \mathbb{R} \times \mathbb{R} : \ f((x, y)) \in \{0\}\} \\ &= \{(x, y) \in \mathbb{R} \times \mathbb{R} : \ 2x - 5y = 0\} \\ &= \{(x, y) \in \mathbb{R} \times \mathbb{R} : \ y = \tfrac{2x}{5}\} \\ &= \left\{\left(x, \tfrac{2x}{5}\right) : \ x \in \mathbb{R}\right\}. \end{aligned}$$

Example. *Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $f((x, y)) = 2x - 5y$. Let V be the open interval $V = (0, 1)$ in $\mathbb{R}$. Then*

$$\begin{aligned} f^{-1}[V] &= \{(x, y) \in \mathbb{R} \times \mathbb{R} : \ f((x, y)) \in V\} \\ &= \{(x, y) \in \mathbb{R} \times \mathbb{R} : \ 2x - 5y \in (0, 1)\} \\ &= \{(x, y) \in \mathbb{R} \times \mathbb{R} : \ 0 < 2x - 5y < 1\} \\ &= \left\{(x, y) \in \mathbb{R} \times \mathbb{R} : \ \tfrac{5}{2}y < x < \tfrac{5}{2}y + \tfrac{1}{2}\right\}. \end{aligned}$$

Hence, we can write

$$f^{-1}[V] = \left\{\left(\tfrac{5}{2}y + \varepsilon, y\right) : \ \epsilon, y \in \mathbb{R}, \ 0 < \varepsilon < \tfrac{1}{2}\right\}.$$

Example. *Define* $g : \mathbb{R} \to \mathbb{R}$ *by* $g(x) = |x^3|$. *Take* $V = [4, \infty)$. *Then*

$$\begin{aligned} g^{-1}[V] &= \{x \in \mathbb{R} : \ g(x) \in V\} \\ &= \{x \in \mathbb{R} : \ |x^3| \in [4, \infty)\} \\ &= \{x \in \mathbb{R} : \ (x^3 \geq 4) \ \vee \ (-x^3 \geq 4)\} \\ &= \{x \in \mathbb{R} : \ (x^3 \geq 4) \ \vee \ (x^3 \leq -4)\} \\ &= \{x \in \mathbb{R} : \ (x \geq \sqrt[3]{4}) \ \vee \ (x \leq \sqrt[3]{-4})\} \\ &= (-\infty, \sqrt[3]{-4}\ ] \cup [\sqrt[3]{4}, \infty). \end{aligned}$$

We have the following theorem.

Theorem 4.10. *Let* $f : X \to Y$, $U \subseteq X$ *and* $V \subseteq Y$. *Then*

(i) $f[f^{-1}[V]] \subseteq V$, *and for* f *surjective, we have* $f[f^{-1}[V]] = V$.

(ii) $U \subseteq f^{-1}[f[U]]$, *and for* f *injective, we have* $U = f^{-1}[f[U]]$.

Proof. We will prove part (i) and leave part (ii) as Exercise 3.8 b).
If $V = \emptyset$, then $f^{-1}[V] = \emptyset$ and $f[f^{-1}[V]] = \emptyset = V$. So suppose $V \neq \emptyset$ and choose $y \in f[f^{-1}[V]]$. Then $y = f(w)$ for some $w \in f^{-1}[V]$. By the definition of $f^{-1}[V]$, we have $f(w) \in V$. Hence, $y = f(w) \in V$. Since y is arbitrary, this shows that every element of $f[f^{-1}[V]]$ lies in V, that is $f[f^{-1}[V]] \subseteq V$.

Now suppose that f is surjective. We need to show that $V \subseteq f[f^{-1}[V]]$. So suppose that $v \in V$. Since f is surjective, there exists an $x \in X$ such that $f(x) = v$. Then $f(x) \in V$, and it follows that $x \in f^{-1}[V]$. Hence, $v = f(x) \in f[f^{-1}[V]]$. Since v was arbitrary, this shows that $V \subseteq f[f^{-1}[V]]$. Summarising, we have that $f[f^{-1}[V]] = V$ whenever f is surjective. □

We have the following theorem.

Theorem 4.11. *Suppose that* $f : X \to Y$ *and* $U, V \subseteq Y$. *Then*

(i) $f^{-1}[U \cap V] = f^{-1}[U] \cap f^{-1}[V]$.

(ii) $f^{-1}[U \cup V] = f^{-1}[U] \cup f^{-1}[V]$.

Proof. We will prove part (i) and leave part (ii) as Exercise 3.8 a).
We have

$$\begin{aligned} f^{-1}[U \cap V] &= \{x \in X : \ f(x) \in U \cap V\} \\ &= \{x \in X : \ (f(x) \in U) \wedge (f(x) \in V)\}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} f^{-1}[U] \cap f^{-1}[V] &= \{x \in X : \ f(x) \in U\ \} \cap \{x \in X : \ f(x) \in V\} \\ &= \{x \in X : \ (f(x) \in U) \wedge (f(x) \in V)\}. \end{aligned}$$

Therefore, $f^{-1}[U \cap V] = f^{-1}[U] \cap f^{-1}[V]$. □