8 Cardinality

Definition 8.1. *We say that two nonempty sets A and B have the same* ***cardinality*** *if there exists a bijective map $f : A \to B$, and we write $|A| = |B|$.*

We have the following properties.

- If there is an injective map $f : X \to Y$, then $|X| \leq |Y|$.
- If there is an injection from X into Y but no bijection between X and Y, we write $|X| < |Y|$.
- When $A = \emptyset$, we set $|A| = 0$.
- Let $n \in \mathbb{N}$. If $f : \{1, 2, \dots, n\} \to A$ is bijective, then $|A| = n$, and we say A has n elements. Further, we can enumerate the elements of A as $a_1, a_2, \dots, a_n$ where $a_i = f(i)$ for all $1 \leq i \leq n$, and since f is injective, $a_i = a_j$ if and only if $i = j$.

Definition 8.2. *When $|A| \in \mathbb{N}_0$, we say A is a* ***finite set****.*

Definition 8.3. *When A is not a finite set, we say A is an* ***infinite set****.*

Let A and B be sets such that $B \subseteq A$. We have the following properties.

- Let A be a finite set such that $|A| = n$, for some $n \in \mathbb{N}_0$. Then $|B| = m$ where $m \leq n$, for some $m \in \mathbb{N}_0$, and $A = B$ if and only if $m = n$. Hence, if A is finite, then B is finite.
- If B is infinite, then A is infinite. (This is the contrapositive of the case above)
- We have that $\mathbb{N}$ is infinite.
- We have $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}_+| = |\mathbb{Q}|$, but $|\mathbb{N}| < |\mathbb{R}|$.

We have the following proposition.

Proposition 8.4. *Let X be a set and let $A, B \subseteq X$ be nonempty finite sets with $A \cap B = \emptyset$. Then $|A \cup B| = |A| + |B|$.*

Proof. Let $m, n \in \mathbb{N}$ such that $|A| = m$ and $|B| = n$. Then

$$A = \{a_1, a_2, \dots, a_m\}$$

where $a_i = a_j$ only if $i = j$, for integers $1 \leq i, j \leq m$. Similarly, we have

$$B = \{b_1, b_2, \dots, b_n\}$$

where $b_i = b_j$ only if $i = j$, for integers $1 \leq i, j \leq n$. Since $A \cap B = \emptyset$, we know that for any integers i, j with $1 \leq i \leq m$ and $1 \leq j \leq n$, we have $a_i \neq b_j$. Now, define $f : \{1, 2, \dots, m + n\} \to A \cup B$ by

$$f(k) = \begin{cases} a_k & \text{if } 1 \leq k \leq m, \\ b_{k-m} & \text{if } m < k \leq m + n. \end{cases}$$

In Exercise 5.6, one shows that f is bijective. □

Definition 8.5. *We say a set X is* ***countable*** *if there exists a bijective function $f : \mathbb{N} \to X$, or equivalently, if there is a bijective function $g : X \to \mathbb{N}$.*

Remark. *Note that some texts in literature say that a set if countable if it is finite or if there exists a bijective function $f : \mathbb{N} \to X$, and when there exists a bijective function $f : \mathbb{N} \to X$, these texts say X is countably infinite.*

Remark. *Combining Definition 8.1 and Definition 8.5, we have that if X is a countable set, then $|X| = |\mathbb{N}|$. So all countable sets are infinite and have the same cardinality as $\mathbb{N}$.*

Suppose that X is a countable set. Then by definition, there exists a bijective map $f : \mathbb{N} \to X$. Hence, we can enumerate the elements of X as $x_1, x_2, x_3, \ldots$ where $x_i = f(i)$ for $i \in \mathbb{N}$.

Example. *The set of positive even integers is countable.*

Proof. Let
$$A = \{2x : x \in \mathbb{N}\},$$
and define $f : \mathbb{N} \to A$ by $f(x) = 2x$, for all $x \in \mathbb{N}$. To see that f is injective, suppose that $x, y \in \mathbb{N}$ such that $f(x) = f(y)$. Then $2x = 2y$, so $x = y$, showing that f is injective.

To see that f is surjective, take $a \in A$. Then $a = 2x$, for some $x \in \mathbb{N}$, and hence $a = 2x = f(x)$. Hence, f is surjective. This shows that f is bijective. Therefore, A is countable. □

Similarly, the set of odd positive integers, $\{2x - 1 : x \in \mathbb{N}\}$, can be shown to be countable.

We have the following results.

Proposition 8.6. *Let X and Y be two sets such that $|X| = |Y|$. If X is countable, then Y is countable.*

Proof. Let $|X| = |Y|$ and suppose that X is countable. Then there are bijections $f : X \to Y$ and $g : \mathbb{N} \to X$. It follows that $f \circ g : \mathbb{N} \to Y$ is bijective, so Y is countable. □

Theorem 8.7. *Suppose $f : X \to Y$ is injective and $A \subseteq X$. Then $|A| = |f[A]|$.*

Proof. Let $B = f[A]$. Define $g : A \to B$ by $g(a) = f(a)$, for all $a \in A$. By the definition of B, we have that $B = f[A] = g[A]$, so g is surjective. Now, suppose that $a, a' \in A$ are such that $g(a) = g(a')$. Then $f(a) = f(a')$, and since f is injective, this means that $a = a'$. Hence, g is injective. It follows that g is bijective, so $|A| = |g[A]|$. We also know that $g[A] = B = f[A]$, so $|A| = |g[A]| = |f[A]|$. □

We have the following two theorems whose proofs are beyond the scope of this course.

Theorem 8.8. *Every infinite set contains a countable subset.*

Theorem 8.9 (Cantor-Schröder-Bernstein Theorem)**.** *If X, Y are sets with $|X| \leq |Y|$ and $|Y| \leq |X|$ then $|X| = |Y|$.*

Remark. *In some texts, Theorem 8.9 is called the Cantor-Bernstein Theorem or the Schöder-Bernstein Theorem. An interesting proof of this theorem due to Halmos can be found in the book Algebra by Pierre Grillet, which is available as an electronic book from the University of Bristol library.)*

Corollary 8.10. *Let X be a subset of $\mathbb{N}$. Then X is either finite or countable.*

Proof. If X is finite then we are done. So suppose X is infinite. We have an injective map $g : X \to \mathbb{N}$ given by $g(x) = x$, so $|X| \leq |\mathbb{N}|$. On the other hand, we know that X contains a countable subset A. Hence, there exists a bijective map $h : \mathbb{N} \to A$. Now, define $f : \mathbb{N} \to X$ by $f(n) = h(n)$, for all $n \in \mathbb{N}$. Then f is an injective map from $\mathbb{N}$ into X. Hence, $|\mathbb{N}| \leq |X|$, and so by the Cantor-Bernstein Theorem, we have that $|X| = |\mathbb{N}|$. Then f must be bijective. It follows that X is countable. □

The next result is very useful when proving that a set is countable.

Corollary 8.11. *Suppose X is an infinite set. Then X is countable if and only if there exists an injective map $f : X \to \mathbb{N}$.*

Proof.

($\Rightarrow$) First, we will show that if there exists an injective map $f : X \to \mathbb{N}$, then X is countable. So suppose there exists such a map. Then $|X| = |f[X]|$, and $f[X]$ is a subset of $\mathbb{N}$. Since X is not finite and $|X| = |f[X]|$, $f[X]$ is not finite. Hence, $f[X]$ is countable. Therefore, we must have that X is countable.

($\Leftarrow$) Second, we will show that if X is countable, then there exists an injective map $f : X \to \mathbb{N}$. So suppose that X is countable. Then there exists a bijective map $g : \mathbb{N} \to X$. Since g is bijective, then g^{-1} exists. Setting $f = g^{-1}$, we have that $f : X \to \mathbb{N}$ is bijective and hence injective.

Summarising, X is countable if and only if there exists an injective map $f : X \to \mathbb{N}$. □

The proof of the following proposition is left as an exercise.

Proposition 8.12. *Suppose that X is a countable set.*

(i) if $A \subseteq X$, then A is either finite or countable.

(ii) if $A \subseteq X$ and A is finite, then $X \setminus A$ is countable.

(iii) there exists $B \subseteq X$ such that B and $X \setminus B$ are countable.

(iv) if $f : C \to X$ is injective, then C is either a finite set or a countable set.

Proof. Exercise 5.7. □

We have the following theorem.

Theorem 8.13. *Let X be a set and let $A, B \subseteq X$. Suppose that A is a countable set and that B is a nonempty and finite set with $A \cap B = \emptyset$. Then $A \cup B$ is countable.*

Proof. Since A is countable, there exists an injective map $f : A \to \mathbb{N}$. Since B is finite we have that $B = \{b_1, \ldots, b_m\}$, for some $m \in \mathbb{N}$, where $|B| = m$. Now, define $g : A \cup B \to \mathbb{N}$ by

$$g(x) = \begin{cases} i, & \text{if } x = b_i, \\ f(x) + m, & \text{if } x \in A. \end{cases}$$

We claim that g is injective. To see this, take $x, y \in A \cup B$ such that $x \neq y$. If $x, y \in B$, then $x = b_i$ and $y = b_j$ for some $i, j \in \mathbb{N}$ such that $i \leq m$, $j \leq m$ with $i \neq j$. Hence, we have that $g(x) = i \neq j = g(y)$. If $x \in B$ and $y \in A$, then $x = b_i$ for some $i \in \mathbb{N}$ with $i \leq m$.

Hence, we have that $g(x) = i < m + 1 \leq g(y) + m + 1$. If $x, y \in A$, then $f(x) \neq f(y)$ since $x \neq y$ and f is injective. So $g(x) = f(x) + m + 1 \neq f(y) + m + 1 = g(y)$. It follows that g is injective.

Since $A \subseteq A \cup B$ and A is infinite, $A \cup B$ is infinite. Since $g : A \cup B \to \mathbb{N}$ is injective and $A \cup B$ is infinite, $A \cup B$ is countable. □

We have the following theorem.

Theorem 8.14. *We have that $\mathbb{N} \times \mathbb{N}$ is countable.*

Proof. Since $\{(x, 1) : x \in \mathbb{N}\}$ is an infinite subset of $\mathbb{N} \times \mathbb{N}$, we have that $\mathbb{N} \times \mathbb{N}$ is infinite. Now, we arrange the elements of $\mathbb{N} \times \mathbb{N}$ in a grid:

$$\begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & \cdots \\ (2,1) & (2,2) & (2,3) & (2,4) & \cdots \\ (3,1) & (3,2) & (3,3) & (3,4) & \cdots \\ (4,1) & (4,2) & (4,3) & (4,4) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{matrix}$$

We order the elements of this grid along the cross-diagonals:

$$(1,1);\ (1,2),\ (2,1);\ (1,3),\ (2,2),\ (3,1);\ldots$$

We want to define a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that

$$\begin{aligned} f((1,1)) &= 1 \\ f((1,2)) &= 2 \\ f((2,1)) &= 3 \\ f((1,3)) &= 4 \\ f((2,2)) &= 5 \\ f((3,1)) &= 6 \\ &\vdots \end{aligned}$$

We have that the kth cross-diagonal contains the pairs $(1, k), (2, k-1), (3, k-2), \ldots, (k, 1)$, a total of k pairs. Hence, the number of pairs in the first $k - 1$ cross-diagonals is given by

$$1 + 2 + 3 + \cdots + (k-1) = \frac{(k-1)k}{2}.$$

Therefore, we define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by

$$f((i, k + 1 - i)) = \frac{(k-1)k}{2} + i,$$

for all $i, k \in \mathbb{N}$.

To show that f is injective, suppose that $x, y \in \mathbb{N} \times \mathbb{N}$ such that $f(x) = f(y)$. Then there exist $i, k \in \mathbb{N}$ such that $i \leq k$ and $x = (i, k + 1 - i)$. Now, assume that y is also on the kth cross-diagonal. Then there exists $j \in \mathbb{N}$ such that $j \leq k$ and $y = (j, k + 1 - j)$. Hence, we have

$$\frac{(k-1)k}{2} + i = f(x) = f(y) = \frac{(k-1)k}{2} + j.$$

Then $i = j$, and it follows that $x = y$.

Now, suppose that y is not on the kth cross-diagonal. Then there exist $j, m \in \mathbb{N}$ such that $j \leq m$ and $y = (j, m+1-j)$. Hence, y is on the mth cross-diagonal where $m \neq k$. Therefore, either $m > k$ or $k > m$. Without loss of generality, assume that $m > k$. Then $m = k + r$ for some $r \in \mathbb{N}$, and we have

$$f(x) = \frac{(k-1)k}{2} + i \leq \frac{(k-1)k}{2} + k.$$

Then

$$\begin{aligned} f(y) &= \frac{(k+r-1)(k+r)}{2} + j \\ &= \frac{(k-1)k}{2} + kr + \frac{(r-1)r}{2} + j \\ &\geq \frac{(k-1)k}{2} + k + 1 \end{aligned}$$

since $kr \geq k$, $r(r-1) \geq 0$, and $j \geq 1$. Therefore, $f(x) \neq f(y)$, contradicting the assumption that $f(x) = f(y)$. Hence, if $f(x) = f(y)$ then x and y are on the same cross-diagonal and $x = y$. It follows that f is injective. □

Suppose X, Y are countable. Then $X \times Y$ is infinite: If we choose $y_0 \in Y$ and define $f : X \times \{y_0\} \to X$ by $f((x, y_0)) = x$, for all $(x, y_0) \in X \times \{y_0\}$, then on can easily show that f is bijective, so $|X \times \{y_0\}| = |X|$, and hence $X \times \{y_0\}$ is countable. Since $X \times \{y_0\} \subseteq X \times Y$, we have that $X \times Y$ is infinite.

We have the following corollaries.

Corollary 8.15. *We have that $\mathbb{Q}_+$ and $\mathbb{Q}$ are countable.*

Proof. Exercise 6.1. □

Remark. *Since $\mathbb{Z}$ is an infinite subset of $\mathbb{Q}$ and $\mathbb{Q}$ is countable, we must have that $\mathbb{Z}$ is countable.*

Corollary 8.16. *Let X and Y be countable sets. Then*

(i) $X \times Y$ is countable.

(ii) if $X \cap Y = \emptyset$, then $X \cup Y$ is countable.

Proof. Exercise 6.2. □

We have the following corollary.

Corollary 8.17. *Let $\{A_n : n \in \mathbb{N}\}$ be a (countable) collection of countable sets that are pairwise disjoint. Then $\bigcup_{n \in \mathbb{N}} A_n$ is countable.*

Proof. First, note that since A_1 is countable, we have that it is infinite. Since $A_1 \subseteq \bigcup_{n \in \mathbb{N}} A_n$, we know that $\bigcup_{n \in \mathbb{N}} A_n$ is also infinite. Further, we have seen that if X is a countable set, Y is an infinite set, and there exists an injective map $g : Y \to X$, then Y is also countable. Hence, in order to prove that $\bigcup_{n \in \mathbb{N}} A_n$ is countable, we will prove there exists an injective function $g : \bigcup_{n \in \mathbb{N}} A_n \to \mathbb{N} \times \mathbb{N}$.

For each $n \in \mathbb{N}$, enumerate the elements of A_n as $a_{n1}, a_{n2}, a_{n3}, \ldots$. Now, define $g : \bigcup_{n=1}^{\infty} A_n \to \mathbb{N} \times \mathbb{N}$ by

$$g(a_{mk}) = (m, k),$$

for all $a_{mk} \in \bigcup_{n=1}^{\infty} A_n$. To see that g is injective, suppose that $g(a_{mk}) = g(a_{st})$, for some $m, k, s, t \in \mathbb{N}$. Then $(m, k) = (s, t)$, so $m = s$, $k = t$, and hence $a_{mk} = a_{st}$. It follows that g is injective. Hence, we have an injective function from $\bigcup_{n=1}^{\infty}$ into a countable set. Since $\bigcup_{n=1}^{\infty} A_n$ is infinite, it follows that $\bigcup_{n=1}^{\infty} A_n$ is countable. □

Remark. *The result of Corollary 8.17 still holds in the case when $\{A_n : n \in \mathbb{N}\}$ is a (countable) collection of countable sets which are **not** pairwise disjoint. In this case, one shows in Exercise 6.3 that there exists a (countable) collection $\{B_n : n \in \mathbb{N}\}$ of countable sets which are pairwise disjoint such that $\bigcup_{n\in\mathbb{N}} B_n = \bigcup_{n\in\mathbb{N}} A_n$.*