9 Uncountable Sets and Power Sets

9.1 Uncountable Sets

Definition 9.1. *A set X is called* ***uncountable*** *if it is infinite but not countable.*

We want to show that $\mathbb{R}$ is uncountable. To do this we will show that the interval $(0,1) = \{x \in \mathbb{R} : \ 0 < x < 1 \ \}$ is uncountable. As an exercise, one can then show that there exists a bijection between the interval $(0,1)$ and $\mathbb{R}$.

Now, assume that every real number between 0 and 1 has a decimal expansion of the form

$$0.a_1a_2a_3\cdots = \sum_{k\in\mathbb{N}} a_k 10^{-k}$$

where a_k is an integer such that $0 \le a_k \le 9$ for each $k \in \mathbb{N}$. Note that

$$\begin{aligned} 0.999\cdots &= \sum_{k\in\mathbb{N}} 9 \cdot 10^{-k} \\ &= 9\left(\frac{1/10}{1-1/10}\right) \\ &= 1 \end{aligned}$$

since $\sum_{k\in\mathbb{N}} 10^{-k}$ is a convergent geometric series. Consequently, if there exists some $N \in \mathbb{N}$ such that $a_N \ne 9$ and $a_n = 9$ for all $n \in \mathbb{N}$ with $n > N$, we have that

$$0.a_1a_2a_3\cdots = 0.a_1a_2\cdots a_{N-1}b_N$$

where $b_N = a_N + 1$. We will assume the result that for every $\alpha \in \mathbb{R}$ with $0 < \alpha < 1$, there is a unique way to write α as $0.a_1a_2a_3\cdots$ where a_k is an integer such that $0 \le a_k \le 9$ for each $k \in \mathbb{N}$ and for all $N \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $n \ge N$ and $a_n \ne 9$.

We have the following theorem.

Theorem 9.2. *The interval $(0,1) = \{x \in \mathbb{R} : 0 < x < 1\}$ is uncountable.*

Proof. (Cantor's diagonalisation argument). We know that the interval $(0,1)$ is infinite, since $f : \mathbb{N} \to (0,1)$ defined by $f(k) = 10^{-k}$, for all $k \in \mathbb{N}$, is easily shown to be injective.

To the contrary, suppose that $(0,1)$ is countable. Hence, we can enumerate the elements of $(0,1)$ as $\alpha_1, \alpha_2, \alpha_3, \ldots$. We write each α_k as a decimal expansion as described above, that is

$$\alpha_k = 0.a_{k1}a_{k2}a_{k3}\cdots$$

where a_{ki} is an integer such that $0 \le a_k \le 9$ for each $k \in \mathbb{N}$. Further, we assume that, for all $N \in \mathbb{N}$, there exists an $n \in \mathbb{N}$ such that $n > N$ and $a_n \ne 9$. Now, for each $k \in \mathbb{N}$, set

$$b_k = \begin{cases} 1, & \text{if } a_{kk} \ne 1; \\ 2, & \text{if } a_{kk} = 1, \end{cases}$$

and set $\beta = 0.b_1b_2b_3\cdots$. Then $\beta \in \mathbb{R}$ with $0 < \beta < 1$ and for all $N \in \mathbb{N}$, there exists an $n \in \mathbb{N}$ such that $n > N$ and $b_n \ne 9$. Hence by assumption, $\beta = \alpha_m$ for some $m \in \mathbb{N}$. But $b_m \ne a_{mm}$, which contradicts the uniqueness of the representation of β as a decimal expansion not ending in an infinite sequence of 9s. Hence, we must have that $(0,1)$ is uncountable. ☐

Remark. *Suppose that we have* $m \in \mathbb{N}$ *where* a_m *is an integer such that* $0 \leq a_m \leq 9$ *(not all* 0*) for each* m*, and*

$$\alpha = 0.a_1a_2\cdots a_ma_1a_2\cdots a_ma_1a_2\cdots a_m\cdots = 0.\overline{a_1a_2\cdots a_m}.$$

Then α *is a rational number. This follows since if* $b = \sum_{k=1}^{m} a_k \cdot 10^{m-k}$*, then* $b \in \mathbb{N}$ *and*

$$\begin{aligned}\alpha &= \sum_{n\in\mathbb{N}} b \cdot 10^{-mn}\\ &= b\left(\frac{10^{-m}}{1-10^{-m}}\right)\\ &= \frac{b}{10^m - 1}.\end{aligned}$$

Note that the map $g : (0,1) \to \mathbb{R}$ given by $g(x) = x$, for all $x \in (0,1)$ is injective, so $|(0,1)| \leq |\mathbb{R}|$. Since $|\mathbb{N}| < |(0,1)|$, we have that $|\mathbb{N}| < |\mathbb{R}|$, that is $\mathbb{R}$ is uncountable. The following corollary, whose proof is left as an exercise, shows that $|(0,1)| = |\mathbb{R}|$. This is another way to argue that $\mathbb{R}$ is uncountable.

Corollary 9.3. *There exists a bijection between the interval* $(0,1)$ *and* $\mathbb{R}$*.*

Proof. Exercise 6.4. □

9.2 Power Sets

Definition 9.4. *Let* A *be a set and define* $\mathscr{P}(A) = \{C : C \subseteq A\}$*. Then* $\mathscr{P}(A)$ *is called the* ***power set*** *of* A*.*

Example. *We have* $\mathscr{P}(\emptyset) = \{\emptyset\}$*, so* $|\mathscr{P}(\emptyset)| = 1$*.*

Example. *We have that* $\mathscr{P}(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$*, so* $|\mathscr{P}(\{1,2\})| = 4 = 2^2$*.*

Example. *We have that* $\mathscr{P}(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$*. So* $|\mathscr{P}(\{1,2,3\})| = 8 = 2^3$*.*

Note that, for any nonempty set X, we know that $\emptyset, X$ are distinct subsets of X. Hence, we have that $|\mathscr{P}(X)| \geq 2$. We have the following results, whose proofs are left as an exercise.

Theorem 9.5. *Suppose that* A *is a finite set with* $|A| = n$ *for some* $n \in \mathbb{N}_0$*. Then* $|\mathscr{P}(A)| = 2^n$*.*

Proof. Exercise 6.5. □

Proposition 9.6. *Let* A, B *be sets. Then*

(i) $A \subseteq B$ *if and only if* $\mathscr{P}(A) \subseteq \mathscr{P}(B)$*.*

(ii) $\mathscr{P}(A) \cup \mathscr{P}(B) \subseteq \mathscr{P}(A \cup B)$*.*

(iii) $\mathscr{P}(A) \cap \mathscr{P}(B) = \mathscr{P}(A \cap B)$*.*

Proof. Exercise 6.6. □

We have the following theorem.

Theorem 9.7 (Cantor's Theorem). *Let X be a set. Then $|X| < |\mathscr{P}(X)|$.*

Proof. Suppose $X = \emptyset$. Then $|X| = 0 < 1 = |\mathscr{P}(X)|$. So suppose X is nonempty, and define $f : X \to \mathscr{P}(X)$ by $f(x) = \{x\}$, for all $x \in X$. We will show that f is injective. So suppose that $x, x' \in X$ are such that $f(x) = f(x')$. Then $\{x\} = \{x'\}$. Hence, we have that $x_1 = x_2$. Therefore, f is injective, so $|X| \leq |\mathscr{P}(X)|$.

Now, we want to show that there exists no bijection between X and $\mathscr{P}(X)$. To the contrary, suppose there exists such a bijection $g : X \to \mathscr{P}(X)$. Define

$$A = \{x \in X : x \notin g(x)\}.$$

Then A is a subset of X, so $A \in \mathscr{P}(X)$. Further, since g is bijective, there exists some $z \in X$ such that $g(z) = A$. By the definition of g, $z \in A$ if and only if $z \notin g(z) = A$, which is a contradiction since $z \in A$ if and only if $z \notin A$. Therefore, there does not exist a bijective function $g : X \to \mathscr{P}(X)$. It follows that $|X| < |\mathscr{P}(X)|$. □