Writing Mathematical Proofs

- Exercise Solutions -

1 Writing mathematics - Exercise Solutions

1. Find the mistake in the following argument: Let $a = b$. Then

$$\begin{aligned} ab &= a^2 \\ a^2 + ab &= a^2 + a^2 \\ a^2 + ab &= 2a^2 \\ a^2 + ab - 2ab &= 2a^2 - 2ab \\ a^2 - ab &= 2a^2 - 2ab \quad (1) \\ 1(a^2 - ab) &= 2(a^2 - ab). \quad (2) \end{aligned}$$

   Dividing both sides of the equation by $a^2 - ab$, we get

$$1 = 2.$$

   Solution: Since $a = b$, Equation (1) essentially says that $0 = 0$ and Equation (2) says that $1 \cdot 0 = 0 = 2 \cdot 0$. We then proceed to divide by $a^2 - ab = 0$, but $\frac{0}{0}$ is an undefined quantity!

2. Why do the following arguments fail? Rewrite the arguments accordingly.

   (i) Let $f(x) = 3x$. Then f is not surjective since there is no $x \in \mathbb{N}$ such that $f(x) = 2$.

   (ii) Let $x = \frac{a}{b}$ and $y = \frac{c}{d}$. Then

$$x + y = \frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd}.$$

   It follows that $x + y$ is rational.

   (iii) Suppose $a \mid b$ and $b \mid c$. Then there exist m and n such that $am = b$ and $bn = c$. It follows that $c = bn = (am)n = a(mn)$. Therefore, $a \mid c$.

   Solution:

   (i) The argument fails because we did not specify that f is only defined on the natural numbers. For the argument to become valid, we could write the following:

   Let $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(x) = 3x$, for all $x \in \mathbb{N}$. Then f is not surjective since there is no $x \in \mathbb{N}$ such that $f(x) = 2$.

(ii) The argument fails because we did not define a, b, c and d to be integers such that $b, d \neq 0$. For example, if $a = 5\sqrt{2}, b = 2, c = -\sqrt{2}$ and $d = 2$, then $x+y = \frac{5\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$ which is irrational. For the argument to become valid, we could write the following:

Let $x = \frac{a}{b}$ and $y = \frac{c}{d}$, where a, b, c, and d are integers such that $b, c \neq 0$. Then

$$x + y = \frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd}.$$

Since $ad + cb$ and cd are integers and $bd \neq 0$, it follows that $x + y$ is rational.

(iii) The argument fails because we did not define m and n to be integers. For example, if $m = \frac{1}{2}$ and $b = \frac{2}{3}$, then $c = a(mn) = \frac{a}{3}$ which contradicts that $a \mid c$. For the argument to become valid, we could write the following:

Suppose $a \mid b$ and $b \mid c$. Then there exist integers m and n such that $am = b$ and $bn = c$. It follows that $c = bn = (am)n = a(mn)$. Since mn is an integer, it follows that $a \mid c$.

3. How could you improve the following? Rewrite accordingly.

(i)

$$\begin{aligned} f(x) &= f(y) \\ x^2 &= y^2 \\ x &= y \end{aligned}$$

(ii) The limit of $\frac{1}{x}$ as $x \to \infty = 0$.

(iii) If an integer > 1 is prime, then the only divisors: $p = 1\&p$.

(iv) $\gcd(a, b) = 1 \iff a$ and b have no common divisor.

(v) $5 = 2 \therefore$ contradiction.

Possible solutions:

(i) (a) We have that $f(x) = f(y)$ if and only if $x^2 = y^2$. Since both x and y are positive, it follows that $x^2 = y^2$ if and only if $x = y$.

(b) We have that $f(x) = f(y)$ if and only if $x^2 = y^2$. Taking square roots, we obtain that $x = \pm y$. Then $x^2 = y^2$ does **not** imply that $x = y$.

(ii) (a) The limit of $\frac{1}{x}$ as x goes to ∞ is equal to 0.

(b) The limit of $\frac{1}{x}$ as $x \to \infty$ is equal to 0.

(c) We have that $\lim\limits_{x\to\infty} \frac{1}{x} = 0$.

(iii) If p is prime, then the only divisors of p are 1 and p.

Note: If p is prime, this implies that p is an integer greater than 1. *Remember to keep it simple!*

(iv) (a) We have that $\gcd(a, b) = 1$ if and only if a and b have no common divors.

(b) We have that a and b are co-prime if and only if a and b have no common divisors.

(v) Since we obtain that $5 = 2$, we have a contradiction.

4. Cookie competition time! Get in teams and rewrite the proof in Figure 1 to show that $d : \mathbb{R} \to \mathbb{R}$ defined by $d(x, y) = |x - y|$ is a metric, for all $x, y \in \mathbb{R}$.

   Hint: There are four parts to the proof.

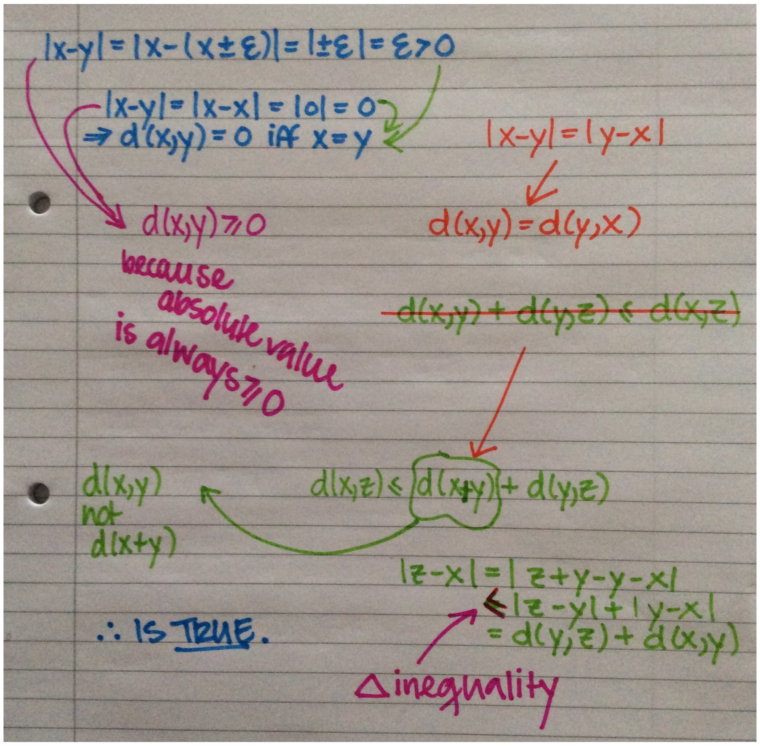

Figure 1

Solution:

Proof. Suppose $x, y \in \mathbb{R}$. To show that $d : \mathbb{R} \to \mathbb{R}$ defined by $d(x, y) = |x - y|$ is a metric, for all x, y, we need to show that d satisfies the following criteria:

(i) $d(x, y) \geqslant 0$.

(ii) $d(x, y) = 0$ if and only if $x = y$.

(iii) $d(x, y) = d(y, x)$.

(iv) $d(x, z) \leqslant d(x, y) + d(y, z)$.

(i) We have that $d(x, y) = |x - y| \geqslant 0$, for all x, y.

(ii) Suppose that $x = y$. Then $d(x, y) = d(x, x) = |x - x| = 0$, for all x, y. Similarly, suppose that $d(x, y) = 0$, for all x, y. Then $0 = |x - x| = |y - y| = d(x, x) = d(y, y)$. It follows that $x = y$.

(iii) We have that $d(x, y) = |x - y| = |-(x - y)| = |y - x| = d(y, x)$, for all x, y.

(iv) Suppose $z \in \mathbb{R}$. Then

$$\begin{aligned} d(x,z) &= |x-z| \\ &= |x-y+y-z| \\ &\leqslant |x-y|+|y-z| \qquad \text{[by the triangle inequality]} \\ &= d(x,y)+d(y,z), \end{aligned}$$

for all x, y, z.

Since d satisfies all of the criteria (i) through (iv), it follows that d is a metric. □

2 Mathematical Logic - Exercise Solutions

1. Determine which of the following sentences are statements. If they are statements, then determine their truth value. If they are not statements, then explain why.

   (i) All lecturers have short hair.
   (ii) Listen to drum and base!
   (iii) Gabi has a yellow iPad.
   (iv) Last year, Steffi ate 200 chocolate cookies.
   (v) This statement is false.
   (vi) The train from Exeter to London takes one hour.
   (vii) Studying mathematics is fun.

   Solution:

   (i) This sentence is a false statement. Certainly, there exists at least one lecturer in the world who has long hair.
   (ii) This sentence is not a statement. It is rather a command!
   (iii) This is a false statment. There are no yellow iPads. Perhaps there are yellow iPad covers but no iPads that are yellow on their own. Therefore, no matter who Gabi is, this will always be a false statement.
   (iv) This sentence is a statement. Even though we cannot determine the truth value of this sentence, we do know that it is either false or true that Steffi ate 200 chocolate cookies last year. It shall forever be a secret.
   (v) Suppose the sentence is a true statement. Then it is true that *the statement is false* which is a contradiction. Now, suppose the sentence is a false statement. Then it is false that *the statement is false*. In other words, the sentence is a true statement which is a contradiction. Therefore, we cannot determine the truth value of the sentence, so it is not a statement.
   (vi) This sentence is definitely a false statement. If only!
   (vii) This sentence is not a statement. It is rather an opinion which people may or may not share.

2. Negate the following sentences:

   (i) Either days are longer in summer or in winter.
   (ii) Days are shorter in summer and in winter.
   (iii) Either days are shorter in summer or longer in winter.
   (iv) Days are longer in summer and shorter in winter.

   Solution: Recall that

   (I) $\neg(A \wedge B) \equiv (\neg A) \vee (\neg B)$, and
   (II) $\neg(A \vee B) \equiv (\neg A) \wedge (\neg B)$,

   Now, w let P and Q be the following two statements:

P: Days are longer in summer.
Q: Days are shorter in winter.

(i) We have $P \vee (\neg Q)$. By (II), it follows that $\neg(P \vee (\neg Q)) \equiv \neg P \wedge Q$. Hence, the negation is given by: *Days are shorter in summer and in winter.*

(ii) We have $(\neg P) \wedge Q$. By (I), it follows that $\neg((\neg P) \wedge Q) \equiv P \vee (\neg Q)$. Hence, the negation is given by : *Either days are longer in summer or in winter.*

(iii) We have $(\neg P) \vee (\neg Q)$. By (II), it follows that $\neg((\neg P) \vee (\neg Q)) \equiv P \wedge Q$. Hence, the negation is given by: *Days are longer in summer and shorter in winter.*

(iv) We have $P \wedge Q$. By (I), it follows that $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$. Hence, the negation is given by: *Either days are shorter in summer or longer in winter.*

3. Find the contrapositive of the following sentences:

(i) If you like numbers, then you like mathematics.
(ii) If you are naughty, then you don't get cookies.
(iii) If you travel to London by train, then the journey takes at least two hours.
(iv) If you are dehydrated, then you didn't drink enough water.
(v) If you like Star Wars, then you are cool.

Solution: Recall that the contrapositive of $A \Longrightarrow B$ is given by $(\neg B) \Longrightarrow (\neg A)$.

(i) If you don't like mathematics, then you don't like numbers.
(ii) If you're nice, then you get cookies.
(iii) If your journey by train takes less than two hours, then you don't travel to London.
(iv) If you drank enough water, then you are not dehydrated.
(v) If you are not cool, then you don't like Star Wars.

4. By using truth tables, show that (a) and (b) are equivalent.

(i) (a) Either Antonella wears blue shoes, or Antonella wears blue and black shoes.
 (b) Antonella wears blue shoes.

(ii) (a) Either it is the case that if Steffi teaches 'Writing Mathematical Proofs' then you learn about negation, or it is the case that if Steffi teaches 'Writing Matheamtical Proofs' then you learn about negative numbers.
 (b) If Steffi teaches 'Writing Mathematical Proofs', then either she teaches about negation or she teaches about negative numbers.

(iii) (a) We have that $x = 2n$, for some integer n, if and only if x is even.
 (b) We have that $x = 2n + 1$, for some integer n, if and only if x is odd.

(iv) (a) It is not the case that Exeter is in Devon if and only if Devon is in England.
 (b) Exeter is in Devon if and only if Devon is not in England.

Solution:

(i) Let P and Q be the following two statements:

P: Antonella wears blue shoes.
Q: Antonella wears black shoes.

Then (a) is equivalent to $P \vee (P \wedge Q)$, and (b) is equivalent to P. Since we do not know who Antonella is and therefore do not know which colour shoes she wears, we need to consider both 'true' and 'false' for the the truth value of both P and Q. However, we do know that Antonella cannot wear both blue and black shoes at the same time (assuming that she only has pairs of shoes which are either blue or black). Hence, we have the following truth table:

P	Q	$P \wedge Q$	$P \vee (P \wedge Q)$
T	F	F	T
F	T	F	F

It follows that $P \vee (P \wedge Q) \equiv P$.

(ii) Let P, Q and R be the following two statements:

P: Steffi teaches 'Writing Mathematical Proofs'.
Q: You learn about negation in 'Writing Mathematical Proofs'.
R: You learn about negative numbers in 'Writing Mathematical Proofs'.

Then (a) is equivalent to $(P \Rightarrow Q) \vee (P \Rightarrow R)$ and (b) is equivalent to $P \Rightarrow (Q \vee R)$. We do know that both P and Q are true statements and that R is a false statement. Hence, we have the following truth table:

P	Q	R	$P \Rightarrow Q$	$P \Rightarrow R$	$(P \Rightarrow Q) \vee (P \Rightarrow R)$	$Q \vee R$	$P \Rightarrow (Q \vee R)$
T	T	F	T	F	T	T	T

It follows that $(P \Rightarrow Q) \vee (P \Rightarrow Q) \equiv P \Rightarrow (Q \vee R)$.

(iii) Let P and Q be the following two statements:

P: We have that $x = 2n$, for some integer n.
Q: We have that x is even.

Then (a) is equivalent to $P \Leftrightarrow Q$ and (b) is equivalent to $(\neg P) \Leftrightarrow (\neg Q)$. Depending on x, either both P and Q are true or both P and Q are false. Hence, we have the following truth table:

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Leftrightarrow Q$	$\neg P$	$\neg Q$	$(\neg P) \Rightarrow (\neg Q)$	$(\neg Q) \Rightarrow (\neg P)$	$(\neg P) \Leftrightarrow (\neg Q)$
T	T	T	T	T	F	F	T	T	T
F	F	T	T	T	T	T	T	T	T

It follows that $P \Leftrightarrow Q \equiv (\neg P) \Leftrightarrow (\neg Q)$.

(iv) Let P and Q be the two following statements:

P: Exeter is in Devon.
Q: Devon is in England.

Then (a) is equivalent to $\neg(P \Leftrightarrow Q)$ and (b) is equivalent to $P \Leftrightarrow (\neg Q)$. We know that both P and Q are true statements. Hence, we have the following truth table:

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Leftrightarrow Q$	$\neg(P \Leftrightarrow Q)$	$\neg Q$	$P \Rightarrow (\neg Q)$	$(\neg Q) \Rightarrow P$	$P \Leftrightarrow (\neg Q)$
T	T	T	T	T	F	F	F	T	F

It follows that $\neg(P \Leftrightarrow Q) \equiv P \Leftrightarrow (\neg Q)$.

3 Proof Techniques I - Exercise Solutions

1. Let a and b be integers such that $a \neq 0$. Using a direct proof, show that if $a \mid b$, then $a^2 \mid b^2$.
   Solution:

   Proof. Suppose that $a \neq 0$ and b are integers such that $a \mid b$. Then there exists an integer n such that $b = an$. Squaring both sides of the equation, we obtain $b^2 = (an)^2 = a^2n^2$. Since n^2 is an integer, the result follows. $\square$

2. Let n be an integer. Using a direct proof, show that n^3 is even if and only if n is even.

   [*Hint: For two statements A and B, we have that* $A \Leftrightarrow B \equiv (A \Rightarrow B) \cap (B \Rightarrow A)$.]
   Solution:

   Proof. Suppose n is an integer. First, we will show that if n^3 is even, then n is even. Since n^3 is even, there exists an integer m such that $n^3 = 2n$. Since 2 divides n^3, we must have that 2 divides n. It follows that n is even.

   Next, we will show that if n is even, then n^3 is even. Since n is even, there exists an integer m such that $n = 2m$. From, this we obtain that $n^3 = (2m)^3 = 8m^3 = 2(4m^3)$. Therefore, we have that n^3 is even. The result follows. $\square$

3. Let n be an integer. Using a proof by cases, show that $n^2 - 3n + 9$ is odd.
   Solution:

   Proof. (i) Suppose n is an even integer. Then there exists m such that $n = 2m$. Hence, we have
   $$n^2 - 3n + 9 = (2m)^2 - 3(2m) + 9 = 4m^2 - 6m + 9 = 2(2m^2 - 3m + 4) + 1.$$
   Since $2m^2 - 3m + 4$ is an integer, we have that $n^2 - 3n + 9$ is odd.

   (ii) Suppose n is an odd integer. Then there exists m such that $n = 2m + 1$. Hence, we have
   $$n^2-3n+9 = (2m+1)^2-3(2m+1)+9 = (4m^2+4m+1)-(6m+3)+9 = 4m^2-2m+7 = 2(2m^2-m+3)+1.$$
   Since $2m^2 - m + 3$ is an integer, we have that $n^2 - 3n + 9$ is odd. The result follows.

   $\square$

4. Using a proof by contradiction, show that if $a \geqslant 2$ and b are integers, then $a \nmid b$ or $a \nmid (b + 1)$.
   solution

   Proof. Let $a \geqslant 2$ and b be integers. To the contrary, suppose that $a \mid b$ and $a \mid (b + 1)$. Then there exist integers m and n such that $b = am$ and $b + 1 = an$. Hence, we have that
   $$an = b + 1 = am + 1.$$
   Rearranging, we obtain
   $$1 = an - am = a(n - m).$$
   Since $n - m$ is an integer, it follows that $a \mid 1$, but this is a contradiction since $a \geqslant 2$. The result follows. $\square$

5. Using a proof by contradiction, show that there exists no positive integer n such that $2n < n^2 < 3n$.

   Solution:

   Proof. To the contrary, suppose there exists a positive integer n such that $2n < n^2 < 3n$. Since n is non-zero, we can divide the inequality by n to get $2 < n < 3$. But this is a contradiction since n is an integer. The result follows. $\square$

6. Show that if x is a non-zero real number such that $x + \frac{1}{x} < 2$, then $x < 0$ by

   (i) using a direct proof.

   (ii) using a proof by contradiction.

   Solution:

   (i) Suppose x is a non-zero real number such that $x + \frac{1}{x} < 2$. Multiplying both sides of the inequality by x^2, we obtain

   $$\begin{aligned} x^3 + x &< 2x^2 \\ x^3 - 2x^2 + x &< 0 \\ x(x^2 - 2x + 1) &< 0 \\ x(x-1)^2 &< 0. \end{aligned}$$

   Since $(x-1)^2 \geqslant 0$ and $x(x-1)^2 \neq 0$, we must have that $(x-1)^2 > 0$. But $x(x-1)^2 < 0$, so we must have that $x < 0$.

   (ii) To the contrary, suppose that there exists a non-zero real number x such that $x + \frac{1}{x} < 2$ and $x \geqslant 0$. Since x is non-zero, it follows that $x > 0$. Multiplying both sides of $x + \frac{1}{x} < 2$ by x, we obtain

   $$\begin{aligned} x^2 + 1 &< 2x \\ x^2 - 2x - 1 &< 0 \\ (x-1)^2 &< 0, \end{aligned}$$

   which is clearly a contradiction. The result follows.

4 Proof Techniques II - Exercise Solutions

1. Let x and y be integers. By giving either a direct proof or a proof by contrapositive, show that if $x + y$ is odd, then x and y are of the opposite parity.

   Solution:

   Proof. We will give a proof by contrapositive. So suppose x and y are of the same parity. We first consider the case when both x and y are even. Then there exist integers m and n such that $x = 2m$ and $y = 2n$. Hence, we have

   $$x + y = 2m + 2n = 2(m + n).$$

   Since $m + n$ is an integer, we have that $x + y$ is even as required.

   Next, we consider the case when both x and y are odd. Then there exist integers m and n such that $x = 2m + 1$ and $y = 2n + 1$. Hence, we have

   $$x + y = (2m + 1) + (2n + 1) = 2m + 2n + 2 = 2(m + n + 1).$$

   Since $m + n + 1$ is an integer, we have that $x + y$ is even as required. The result follows by contrapositive. □

2. Let n be a natural number. Prove that $(n + 1)^2 - 1$ is even if and only if n is even. You may use any suitable proof technique.

   Solution: We first note that the if and only if statement can be broken down into two smaller substatements:

   (i) If $(n + 1)^2 - 1$ is even, then n is even.
   (ii) If n is even, then $(n + 1)^2 - 1$ is even.

   The contrapositive of Statement (i) is given by

   (i)(C) If n is odd, then $(n + 1)^2 - 1$ is odd.

   For Statement (i), we will give a proof by contrapositive, and for Statement (ii) we will give a direct proof.

   Proof. First, we will show that if $(n + 1)^2 - 1$ is even, then n is even by giving a proof by contrapositive. So suppose n is odd. Then there exists an integer m such that $n = 2m + 1$. Hence, we have

   $$(n + 1)^2 - 1 = ((2m + 1) - 1)^2 - 1 = (2m)^2 - 1 = 4m^2 - 1 = 4m^2 - 2 + 1 = 2(2m^2 - 1) + 1.$$

   Since $2m^2 - 1$ is an integer, it follows that $(n + 1)^2 - 1$ is odd as required.

   Next, we will show that if n is even, then $(n + 1)^2 - 1$ is even. Since n is even, then there exists an integer m such that $n = 2m$. Hence, we have

   $$(n + 1)^2 - 1 = (2m + 1)^2 - 1 = (4m^2 + 4m + 1) - 1 = 4m^2 + 4m = 2(2m^2 + 2m).$$

   Since $2m^2 + 2m$ is an integer, it follows that $(n + 1)^2 - 1$ is even. The result follows. □

3. Using a proof by induction, show that $5 \mid 6^n - 1$, for all natural numbers n.

Solution:

Proof. Let $P(n)$ be the following statement:

$$5 \mid 6^n - 1.$$

We will show that $P(n)$ is a true statement for all $n \in \mathbb{N}$ by giving a proof by induction. First, let us consider $P(1)$. We have that $6^1 - 1 = 5$ and $5 \mid 5$. Therefore, $P(1)$ is a true statement.

Now, suppose that $P(k)$ is a true statement for some natural number k, that is

$$5 \mid 6^k - 1.$$

Hence, there exists an integer m such that $6^k - 1 = 5m$. Rearranging, we have $6^k = 5m + 1$. Then

$$\begin{aligned} 6^{k+1} - 1 &= 6 \cdot 6^k - 1 \\ &= 6(5m+1) - 1 \\ &= 30m + 6 - 1 \\ &= 30m + 5 \\ &= 5(6m+1). \end{aligned}$$

Since $6m + 1$ is an integer, we have that $5 \mid 6^{k+1} - 1$. Hence, if $P(k)$ is true, then $P(k+1)$ is true. It follows that $P(n)$ is true for all natural numbers n by the Principle of Mathematical Induction. $\square$

4. Using a proof by induction, prove that $2^n > n^2$, for all natural numbers n such that $n \geqslant 5$.

Solution:

Proof. Let $P(n)$ be the following statement:

$$2^n > n^2$$

We will show that $P(n)$ is a true statement for all natural numbers $n \geqslant 5$ by giving a proof by induction. First, let us consider $P(5)$. We have that $2^5 = 32 > 25 = 5^2$. Hence, $P(5)$ is a true statement.

Now, suppose that $P(k)$ is a true statement for some natural number $k \geqslant 5$, that is

$$2^k > k^2.$$

Then

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &> 2k^2, && \text{by assumption,} \\ &= k^2 + k^2 \\ &= k^2 + k \cdot k \\ &\geqslant k^2 + 5k, && \text{since } k \geqslant 5, \\ &= k^2 + 2k + 3k \\ &\geqslant k^2 + 2k + 15, && \text{since } k \geqslant 5 \\ &\geqslant k^2 + 2k + 1 \\ &= (k+1)^2, \end{aligned}$$

showing that if $P(k)$ is true, then $P(k+1)$ is true. By the Principle of Mathematical Induction, it follows that $P(n)$ is true for all natural numbers $n \geqslant 5$. $\square$

5. Suppose that $F_1 = F_2 = 1$ and that

$$F_n = F_{n-1} + F_{n-2},$$

for all natural numbers $n \geqslant 3$. This sequence is called the *Fibonacci sequence*. Show that if $3 \mid n$, then $2 \mid F_n$. You may use any suitable proof technique.

Solution:

Proof. Let $P(n)$ be the following statement:

$$2 \mid F_n.$$

We will show that $P(n)$ is a true statement for all $n \in \mathbb{N}$ such that $3 \mid n$ by giving a proof by induction. Since $3 \mid n$, then the smallest natural number for which $P(n)$ has to be true is given by $n = 3$. We have that

$$F_3 = F_2 + F_1 = 1 + 1 = 2.$$

Since $2 \mid 2$, it follows that $P(3)$ is a true statement. Now, suppose that $P(i)$ is true for some natural number $3 \leqslant i \leqslant k$ such that $3 \mid i$. That is, we have that

$$2 \mid F_i.$$

We now consider three cases, according to whether $k+1 = 3m$, $k+1 = 3m+1$, or $k+1 = 3m+2$, for some integer m.

(i) Suppose that $k+1 = 3m$. Then $3 \mid k+1$ and so $3 \nmid k$ and $3 \nmid k-1$. Then, by assumption, we must have that $2 \nmid F_k$ and $2 \nmid F_{k-1}$. Then F_k and F_{k-1} must be odd. Therefore, $F_k + F_{k-1}$ must be even. It follows that $2 \mid F_k + F_{k-1}$, that is $2 \mid F_{k+1}$.

(ii) Suppose that $k + 1 = 3m + 1$, that is $k = 3m$. Then $3 \mid k$, but $3 \nmid k - 1$. Then, by assumption, we must have that $2 \mid F_k$ and $2 \nmid F_{k-1}$. Then F_k must be even and F_{k-1} must be odd. Therefore, $F_k + F_{k-1}$ must be odd. It follows that $2 \nmid F_k + F_{k-1}$, that is $2 \nmid F_{k+1}$

(ii) Suppose that $k + 1 = 3m + 2$, that is $k - 1 = 3m$. Then $3 \mid k - 1$, but $3 \nmid k$. Then, by assumption, we must have that $2 \mid F_{k-1}$ and $2 \nmid F_k$. Then F_{k-1} must be even while F_k must be odd. Therefore, $F_k + F_{k-1}$ must be odd. It follows that $2 \nmid F_k + F_{k-1}$, that is $2 \nmid F_{k+1}$.

Then $P(k + 1)$ is true only if $3 \mid (k + 1)$. By the Strong Principle of Mathematical Induction, we have that $P(n)$ is a true statement for all integers n such that $3 \mid n$.

$\square$

6. Are the following sentences true or false statements? If they are true, explain why. If they are false, give a proof by counterexample.

(i) If n is a natural number and s is irrational, then $\frac{n}{s}$ is irrational.

(ii) The square of a real number is always positive.

(iii) We have that $5 \mid n^5 - n$, for all integers n.

(iv) If x and y are integers of the same parity, then xy and $(x+y)^2$ are of the same parity.

(v) There exist distinct rational numbers a and b such that $(a-1)(b-1) = 1$.

Solution:

(i) This is a true statement.

Proof. To the contrary, suppose that if n is a natural number and s is irrational, then $\frac{n}{s}$ is rational. Then we may write

$$\frac{n}{s} = \frac{a}{b},$$

where a and b are integers such that $b \neq 0$. Rearranging, we obtain

$$s = \frac{bn}{a}.$$

Since bn and $a \neq 0$ are integers, it follows that s is rational, giving a contradiction. $\square$

(ii) This is a false statement.

Proof. We will disprove the statement by giving a counterexample. We have that $x = 0$ is a real number, but $0^2 = 0$ which is non-negative. Therefore, the statement is false. $\square$

Here, note that if we would change the statement to 'always non-negative', then we would have a true statement.

(iii) This is a false statement.

Proof. We will disprove the statement by giving a counterexample. Take the integer $n = -1$. Then $n^5 - n = (-1)^5 - 1 = -1 - 1 = -2$ and $5 \nmid -2$. Therefore, the statement is false. $\square$

Here, note that if we were to change n to be a natural number, then the statement would actually be true and could be proved by induction.

(iv) This is a false statement.

Proof. We will disprove the statement by giving a counterexample. So suppose that both x and y are odd. Then there exist integers m and n such that $x = 2m+1$ and $y = 2n+1$. Then

$$xy = (2m+1)(2n+1) = 4mn + 2m + 2n + 1 = 2(2mn + m + n) + 1.$$

Since $2mn + m + n$ is an integer, we have that xy is odd. However,

$$(x+y)^2 = ((2m+1) + (2n+1))^2 = (2m + 2n + 2)^2 = (2(m+n+1))^2.$$

Since $m+n+1$ is an integer, we have that $2(m+n+1)$, and therefore $(2(m+n+1))^2 = (x+y)^2$ is even. Hence, the statement is false. $\square$

(v) This is a true statement. Since this is an *there exists* statement, the statement only has to be true in one instance. For example, if we take $a = 3$ and $b = \frac{3}{2}$, then $(a-1)(b-1) = (3-1)(\frac{3}{2} - 1) = 2\left(\frac{1}{2}\right) = 1$.