Writing Mathematical Proofs

- Exercises -

1 Writing Mathematics - Exercises

1. Find the mistake in the following argument: Let $a = b$. Then

$$\begin{aligned} ab &= a^2 \\ a^2 + ab &= a^2 + a^2 \\ a^2 + ab &= 2a^2 \\ a^2 + ab - 2ab &= 2a^2 - 2ab \\ a^2 - ab &= 2a^2 - 2ab \\ 1(a^2 - ab) &= 2(a^2 - ab). \end{aligned}$$

   Dividing both sides of the equation by $a^2 - ab$, we get

$$1 = 2.$$

2. Why do the following arguments fail? Rewrite the arguments accordingly.

   (i) Let $f(x) = 3x$. Then f is not surjective since there is no $x \in \mathbb{N}$ such that $f(x) = 2$.

   (ii) Let $x = \frac{a}{b}$ and $y = \frac{c}{d}$. Then

$$x + y = \frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd}.$$

   It follows that $x + y$ is rational.

   (iii) Suppose $a \mid b$ and $b \mid c$. Then there exist m and n such that $am = b$ and $bn = c$. It follows that $c = bn = (am)n = a(mn)$. Therefore, $a \mid c$.

3. How could you improve the following? Rewrite accordingly.

   (i)

$$\begin{aligned} f(x) &= f(y) \\ x^2 &= y^2 \\ x &= y \end{aligned}$$

(ii) The limit of $\frac{1}{x}$ as $x \to \infty = 0$.

(iii) If an integer > 1 is prime, then the only divisors: $p = 1 \& p$.

(iv) $\gcd(a, b) = 1 \iff a$ and b have no common divisor.

(v) $5 = 2 \therefore$ contradiction.

4. Cookie competition time! Get in teams and rewrite the proof in Figure 1 to show that $d : \mathbb{R} \to \mathbb{R}$ defined by $d(x, y) = |x - y|$ is a metric, for all $x, y \in \mathbb{R}$.

   Hint: There are four parts to the proof.

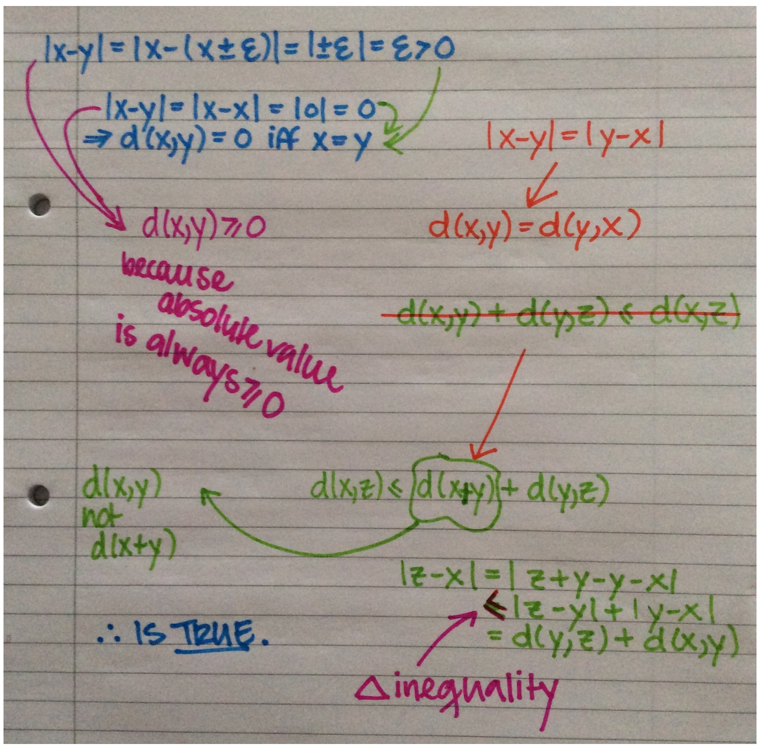

Figure 1

2 Mathematical Logic - Exercises

1. Determine which of the following sentences are statements. If they are statements, then determine their truth value. If they are not statements, then explain why.

   (i) All lecturers have short hair.

   (ii) Listen to drum and base!

   (iii) Gabi has a yellow iPad.

   (iv) Last year, Steffi ate 200 chocolate cookies.

   (v) This statement is false.

   (vi) The train to London takes one hour.

   (vii) Studying mathematics is fun.

2. Negate the following sentences:

   (i) Either days are longer in summer or in winter.

   (ii) Days are shorter in summer and in winter.

   (iii) Either days are shorter in summer or longer in winter.

   (iv) Days are longer in summer and shorter in winter.

3. Find the contrapositive of the following sentences:

   (i) If you like numbers, then you like mathematics.

   (ii) If you are naughty, then you don't get cookies.

   (iii) If you travel to London by train, then the journey takes at least two hours.

   (iv) If you are dehydrated, then you didn't drink enough water.

   (v) If you like Star Wars, then you are cool.

4. By using truth tables, show that (a) and (b) are equivalent.

   (i) (a) Either Antonella wears blue shoes, or Antonella wears blue and black shoes.
       (b) Antonella wears blue shoes.

   (ii) (a) Either it is the case that if Steffi teaches 'Writing Mathematical Proofs' then you learn about negation, or it is the case that if Steffi teaches 'Writing Matheamtical Proofs' then you learn about negative numbers.
       (b) If Steffi teaches 'Writing Mathematical Proofs', then either she teaches about negation or she teaches about negative numbers.

   (iii) (a) We have that $x = 2n$, for some integer n, if and only if x is even.
       (b) We have that $x = 2n + 1$, for some integer n, if and only if x is odd.

   (iv) (a) It is not the case that Exeter is in Devon if and only if Devon is in England.
       (b) Exeter is in Devon if and only if Devon is not in England.

3 Proof Techniques I - Exercises

1. Let a and b be integers such that $a \neq 0$. Using a direct proof, show that if $a \mid b$, then $a^2 \mid b^2$.

2. Let n be an integer. Using a direct proof, show that n^3 is even if and only if n is even.

   [*Hint: For two statements A and B, we have that $A \Leftrightarrow B \equiv (A \Rightarrow B) \cap (B \Rightarrow A)$.*]

3. Let n be an integer. Using a proof by cases, show that $n^2 - 3n + 9$ is odd.

4. Using a proof by contradiction, show that if $a \geqslant 2$ and b are integers, then $a \nmid b$ or $a \nmid (b+1)$.

5. Using a proof by contradiction, show that there exists no positive integer n such that $2n < n^2 < 3n$.

6. Show that if x is a non-zero real number such that $x + \frac{1}{x} < 2$, then $x < 0$ by

   (i) using a direct proof.

   (ii) using a proof by contradiction.

4 Proof Techniques II

1. Let x and y be integers. By giving either a direct proof or a proof by contrapositive, show that if $x+y$ is odd, then x and y are of the opposite parity.

2. Let n be a natural number. Prove that $(n+1)^2-1$ is even if and only if n is even. You may use any suitable proof technique.

3. Using a proof by induction, show that $5 \mid 6^n - 1$, for all natural numbers n.

4. Using a proof by induction, prove that $2^n > n^2$, for all natural numbers n such that $n \geqslant 5$.

5. Suppose that $F_1 = F_2 = 1$ and that
$$F_n = F_{n-1} + F_{n-2},$$
for all natural numbers $n \geqslant 3$. This sequence is called the *Fibonacci sequence*. Show that if $3 \mid n$, then $2 \mid F_n$. You may use any suitable proof technique.

6. Are the following sentences true or false statements? If they are true, explain why. If they are false, give a proof by counterexample.

   (i) If n is a natural number and s is irrational, then $\frac{n}{s}$ is irrational.

   (ii) The square of a real number is always positive.

   (iii) We have that $5 \mid n^5 - n$, for all integers n.

   (iv) If x and y are integers of the same parity, then xy and $(x+y)^2$ are of the same parity.

   (v) There exist distinct rational numbers a and b such that $(a-1)(b-1) = 1$.