Writing Mathematical Proofs

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The main resources for this course are the two following books:

- *Mathematical Proofs* by Chartrand, Polimeni, and Zhang
- *How to Think Like a Mathematician* by Kevin Houston

1 Writing Mathematics

- A mathematical theory is not to be considered complete until you have made it so clear that you can explain it to the first man whom you meet on the street. -
(David Hilbert, mathematician)

But why is this the author's task? In 'How to think like a mathematician', Kevin Houston writes that

(i) 'If the reader has to use their intelligence to work out what was intended, then the student is getting marks because of the reader's intelligence, not their own intelligence.'

(ii) 'This second point is perhaps more important to students. Sorting through a jumble of symbols and half-baked poorly expressed ideas is likely to frustrate and annoy any assessor– not a good recipe for obtaining good marks.'

Example 1 (A naughty proof)**.** We will show that a dog has 9 legs.

Proof. No dog has 5 legs. A dog has 4 more legs than no dog. Therefore, a dog has 9 legs. □

So why is this a naughty proof? Well, letting $d = \textit{dogs}$ and $l = \textit{legs}$, we obtain the equation

$$0d = 5l,$$

but this is equivalent to saying that $l = 0$, so we have no legs to begin with! Of course, this is a contradiction to our initial assumption that no dog has 5 legs.

Example 2 (An equally naughty proof)**.** Do you understand what is being proved in Figure 1? No? I suppose I do not need to say any more.

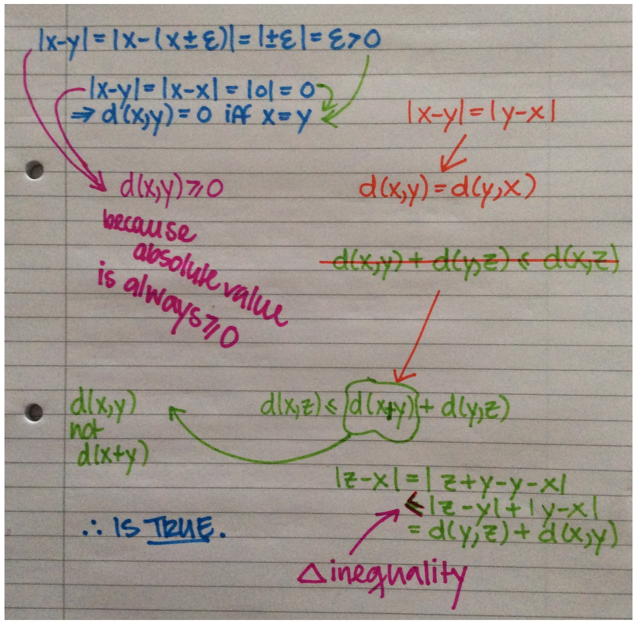

Figure 1: An equally naughty proof.

1.1 How to get started

At the beginning of any proof, clearly state what you are trying to prove and always make sure to define your variables. Readers are not psychic. Well, maybe not all of them. Hence, it is important to explain what you are doing.

You can introduce your argument with, for example,

We will now show that, We will prove that

Similarly, it is important to clearly state your conclusion. You can end your argument with, for example,

Therefore, It follows that, Hence, The result follow.

Throughout your proof, you should justify and connect your statements. The following words may be helpful:

because, as, since, due to, in view of, from, using, by

Example 3. We have the following statement:

$$x^5 > 0.$$

Here, we certainly now that the statement is not true for *all* x. For example, if $x = -1$, then $x^5 = -1 < 0$. Hence, it would be better to add an explanation to the given statement. For example, a possible scenario is that

$$x^5 > 0 \textit{ since } x \textit{ is positive.}$$

Example 4. We have the following statement:

Since f is continuous and $f(2) < 0 < f(5)$, there exists $c \in (2,5)$ such that $f(c) = 0$.

Here you may ask yourself why this statement is true. Are you familiar with the theorem being used? No? Well that is fine, because it is the writer's responsibility to state which theorem he or she is using. It would be better for him or her to write, for example,

Since f is continuous and $f(2) < 0 < f(5)$, there exists $c \in (2,5)$ such that $f(c) = 0$ by the Intermediate Value Theorem.

Connecting phrases are useful in emphasizing that an implication or a deduction is being made.

Example 5. We have the following statement:

$$\frac{\left(\frac{5}{4}\right)}{3} = \frac{5}{12}, \qquad \frac{5}{\left(\frac{4}{3}\right)} = \frac{15}{4},$$

Here, it would be better to write, for example,

$$\textit{We have } \frac{\left(\frac{5}{4}\right)}{3} = \frac{5}{12}, \textit{ but } \frac{5}{\left(\frac{4}{3}\right)} = \frac{15}{4}.$$

Repeating words over and over can be very boring to read. Some synonyms that may be helpful to use for deductions are

hence, so, it follow, therefore, as a result, consequently, thus, accordingly, then.

Some synonyms for explanations are

since, as, because, by, due to, in view of, owing to

Also, for assumptions (also called *hypotheses*), we can use

let, assume that, suppose that.

1.2 Using precise notation

Why is it important to define your notation? The following example shows just how crucial it is.

Example 6. Prove that the $\sqrt{2}$ is irrational.

Proof. (Epic fail.) We will prove that the $\sqrt{2}$ is irrational by giving a proof by contradiction. So suppose $\sqrt{2}$ is rational, and write

$$\sqrt{2} = \frac{a}{b}, \tag{1}$$

where $b \neq 0$. Then squaring both sides of Equation (1) and rearranging, we obtain

$$2b^2 = a^2.$$

It follows that a^2 must be even which implies that a must be even. Hence, we can write $a = 2n$, for some integer n. Substituting this back into Equation 2, we obtain

$$\begin{aligned} 2b^2 &= (2n)^2 \\ 2b^2 &= 4n^2 \\ b^2 &= 2n^2. \end{aligned}$$

It follows that b^2 must be even which implies that b must be even. But this contradicts that $\gcd(a, b) = 1$. The result follows. □

The above proof fails because a crucial piece of information is missing: the assumption that a, b are integers such that $gcd(a, b) = 1$ (and $b \neq 0$). If we rewrote the proof as follows, then the proof would hold.

Proof. (Yay!) We will prove that the $\sqrt{2}$ is irrational by giving a proof by contradiction. So suppose $\sqrt{2}$ is rational, and write

$$\sqrt{2} = \frac{a}{b},$$

where $\frac{a}{b}$ *is in lowest terms, that is* a, b *are integers such that* $\gcd(a, b) = 1$ and $b \neq 0$. Then etcetera, etcetera (see above). □

Also, it is important to be precise when making a statement.

Example 7. Consider the following statement:

$$f(x) = 3x + 2$$

You may ask yourself what x is. Is x a real number? Is x a complex number? Also, is f true for all x, or is f defined on a certain set? Depending on the situation, it would be better to write, for example,

$$f(x) = 3x + 2, \quad \textit{for all } x \in \mathbb{R},$$

or

$$f(x) = 3x + 2, \quad \textit{for } x \in \{1, 2, 3, 4\},$$

depending on the situation.

1.3 Symbols or words?

When writing mathematics it is important to write in complete sentences, and use punctuation. Whenever possible, you should keep it simple and use short words and sentences.

Symbols are basically shorthand notation. For example, a result concerning complex numbers is given

$$\sqrt{-1} = i.$$

If we were to write this in words, it would be less impressive:

The square root of negative one is equal to the imaginary unit.

This seems rather difficult to read. In general, however, it is good to use words. For example, it is always better to write 'therefore' instead of using $\therefore$. Also, it is better to <u>not</u> mix words and symbols.

Example 8. The statement

$$x \textit{ is rational and } y \textit{ is rational} \Longrightarrow x + y \textit{ is rational}$$

is better written as

We have that x is rational and y is rational implies that $x + y$ is rational.

Example 9. The statement

Every integer $\geqslant 2$ is either prime or composite.

is better written as

Every integer greater than 1 is either prime or composite.

A good example to show where it can become very tricky to mix symbols and words in one sentence is given by

$$3 \textit{ divided by } 3 = 1.$$

The sentence reads okay, but here we have the equation $3 = 1$ which is obviously not a true statement, so caution is advised!

Also, it is good practise to **not** start a sentence with a symbol.

Example 10. The statement

X is a finite set.

is better written as

We have that X is a finite set.

2 Mathematical Logic

A **statement** is a sentence which is either true or false. It cannot be both.

Example 11. Are the following statements?

(i) All cats are white.

(ii) All primes greater than 2 are odd.

(iii) Fetch me some water!

(iv) Your lecturer has a big nose.

We have that (i) and (ii) are statements. But what are their truth values? Well, it is not true that all cats are white, so (i) is false. However, it is true that all primes (strictly) greater than 2 are odd. Hence, (ii) is a true statement.

Further, we have that (iii) and (iv) are not statement. Rather, we have that (iii) is a command. And (iv) is an opinion. This is also called a **conditional statement**.

2.1 Logical Equivalence

If we take two statements which are logically equivalent, say $A \equiv B$, then proving A to be true is equivalent to proving B to be true. Similarly, proving B to be true is equivalent to proving A to be true.

Example 12. We have the following logical equivalence: If P and Q are two statements, then $P \implies Q \equiv (\neg P) \vee Q$, where $(\neg P)$ denotes the negation of P. Letting

P: The name of your lecturer is Steffi.
Q: Your lecturer has pink hair.

Then

$P \implies Q$: If your lecturer's name is Steffi, then she has pink hair.

and

$(\neg P) \vee Q$: Either your lecturer's name is not Steffi, or your lecturer has pink hair.

are logically equivalent. Was is the truth value here?

2.2 Negation

The negation of a true statement is always false, and the negation of a false statement is always true. This can be useful in proving statements. For example, previously we wanted to prove that $\sqrt{2}$ is irrational, and we used a proof by contradiction. That is, we used a negation technique. Assuming that our original statement

$$\sqrt{2} \textit{ is irrational}$$

was true, we showed that its negation given by

$$\sqrt{2} \textit{ is rational}$$

was a false statement.

Example 13. The negation of

$$14 \textit{ is composite.}$$

is given by

$$14 \textit{ is prime.}$$

Would you find it easier to prove that 14 *is composite* is a true statement, or that its negation given by 14 *is prime* is a false statement?

Now, let us have a closer look at the negation of statements involving ‘and’ ($\wedge$) and ‘or’ ($\vee$). We have that

(i) $\neg(A \wedge B) \equiv (\neg A) \vee (\neg B)$,

(ii) $\neg(A \vee B) \equiv (\neg A) \wedge (\neg B)$.

Example 14. To convince ourselves of the validity of (i), consider the negation of the statement

Tom is Irish and has red hair.

Considering the negation of this statement, we must have that it is not true that *Tom is Irish and has red hair.* Then either *Tom is not Irish* is a true statement or *Tom does not have red hair* is a true statement. (Remember that this includes the case where both are true!)

Example 15. Similarly, to convince ourselves of the validity of (ii), consider the negation of the statment

Sue wears a blue or a red jacket.

Considering the negation of this statement, we must have that it is not true that *Sue wears a blue or red jacket.* Hence, both *Sue does not wear a blue jacket* and *Sue does not wear a red jacket* are true statements.

We can also check the validity of (i) and (ii) with the help of a truth tables.

A	B	$A \wedge B$	$\neg(A \wedge B)$	$A \vee B$	$\neg(A \vee B)$	$\neg(A)$	$\neg B$	$(\neg A) \vee (\neg B)$	$(\neg A) \wedge \neg(B)$
T	T	T	F	T	F	F	F	F	F
T	F	F	T	T	F	F	T	T	F
F	T	F	T	T	F	T	F	T	F
F	F	F	T	F	T	T	T	T	T

2.3 Contrapositive

We will now study contrapositives. The **contrapositive** for the statement

$$A \implies B$$

is given by

$$(\neg A) \implies (\neg B).$$

Here it is important to recall that $A \implies B$ means that A implies B. However, it does **not** mean that A *causes* B. It merely says that A is a sufficient condition for B.

Example 16. The contrapositive of the statement

If I live in Exeter, then I live in Devon.

is given by

If I do not live in Devon, then I do not live in Exeter.

Example 17. The contrapositive of the statement

If I like cats, then I like dogs.

is given by

If I do not like dogs, then I do not like cats.

Note here that both statements *I like cats* and *I like dogs* are conditional statements– they are dependent on the person making the statement. What would be the truth value in your case?

Example 18. The contrapositive of the statement

If p is a prime greater than 2, then p is odd.

is given by

If p is an even prime, then p is equal to 2.

We have the following theorem:

Theorem 19. *$A \implies B$ is equivalent to $(\neg B) \implies (\neg A)$.*

A	B	$A \implies B$	$(\neg A)$	$(\neg B)$	$(\neg B) \implies (\neg A)$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

By the above theorem, we have that proving the contrapositive of a statement to be true is equivalent to proving the original statement to be true. Indeed, some statements may be easier to prove by using the contrapositive.

Example 20. Suppose that A, B, C and D are sets such that $C \backslash D \subseteq A \cap B$ and that $x \in C$. We want to show that if $x \notin A$, then $x \in D$. Using the contrapositive we will show instead that if $x \notin D$, then $x \in A$.

Proof. Let $x \in C$ and $x \notin D$. Then $x \in C \backslash D$. Since $C \backslash D \subseteq A \cap B$, the result follows. □

A few important important remarks here.

- The contrapositive in not the same as an inverse statement. The inverse statement of $A \implies B$ is given by $(\neg A) \implies (\neg B)$ which is not necessarily a true statement even if $A \implies B$ is a true statement.

  Example 21. The inverse of

  If $x = 4$, then x is even

  is given by

  If $x \neq 4$, then x is odd

  which is not a true statement.

- Do not confuse contrapositive with contradiction. When proving the statement $A \implies B$ by contrapositive, we assume $(\neg A)$ and show that $(\neg B)$. Proving the same statement by contradiction, we assume A and $(\neg B)$ are true, and then show a contradiction. A proof by contrapositive can be in certain case a lot more clear as a proof by contradiction since, here, we would use a direct proof. With a contradiction, it is sometimes less clear what the contradiction is going to be.

  Example 22. Letting A and B be the following two statements

  A: I live in Exeter,
  B: I live in Devon.

We want to prove that

$$A \implies B\text{: If I live in Exeter, then I live in Devon.}$$

For a proof by contrapositive, we would assume that *I do not live in Devon* and then continue to prove that this implies that *I do not live in Exeter*. For a proof by contradiction, we would assume that *I live in Exeter, but I do not live in Devon* and then continue to show that this is a contradiction in itself. Obviously, it will be impossible to live in Exeter without living in Devon!

3 Proof Techniques I

3.1 Direct Proof

Most statements can be broken down into the form 'If A, then B'. To prove that A implies B, the argument can be broken down into smaller substatements. You start with showing that A implies A_1, then A_1 implies A_2, then ..., up until A_n implies B. Each implication $A_1, A_2, \ldots, A_n$ should be easy to follow. This is what we call a direct proof.

Example 23. Let a, b, and c be integers. Show that if a and b are odd, then $ac + cb$ is even.

Proof. Suppose a, b, and c are integers such that a and b are odd. Then there exist integers m and n such that $a = 2n + 1$ and $b = 2m + 1$. Thus

$$ac + cb = c(a + b) = c((2n + 1) + (2m + 1)) = c(2n + 2m + 2) = 2c(m + n + 1).$$

Since $c(m + n + 1)$ is an integer, we have that $ac + cb$ is even, as required. □

Remark 24. A common mistake with direct proofs is to assume what has to be proved. For example, suppose you are trying to prove the following statement:

If a and b are real numbers, then $a^2 + b^2 \geqslant 2ab$.

Incorrect proof. We have

$$\begin{aligned} a^2 + b^2 &\geqslant 2ab \\ a^2 - 2ab + b^2 &\geqslant 0 \\ (a - b)^2 &\geqslant 0. \end{aligned}$$

Since the square of a number is always positive, then $(a - b)^2 \geqslant 2$ is a true statement. It follows that $a^2 + b^2 \geqslant 2ab$. □

The error in the proof is that the conclusion has been assumed to be true and has led to something we know to be true. However, we cannot conclude that a statement is true just because it implies a known truth. This goes back to the fact that A implies B does not mean that A causes B. For example, if A is a false statement and B is a true statement, then $A \Rightarrow B$ is a true statement even though A is a false statement.

However, if you want to find a proof, then it is fine to assume what has to be proved just to see where it leads. This strategy often unlocks the problem and allows us to create a proof. For example, we could use the above incorrect proof and reverse the argument:

$$\begin{aligned} (a - b)^2 &\geqslant 0 \\ a^2 - 2ab + b^2 &\geqslant 0 \\ a^2 + b^2 &\geqslant 2ab. \end{aligned}$$

Thus, it is fine to assume what needs to be proved for exploratory purposes, but when writing the actual proof, it is not.

3.2 Proof by Cases

Let A and B be two sets. One way of proving that $A = B$ is to show that $A \subseteq B$ and $B \subseteq A$. Hence, we have broken down the argument into two cases. Whenever it is convenient (and possible!) to break down an argument into individual cases, we will use a proof by cases.

Example 25. Let A and B be sets. Prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Proof. (i) Suppose $x \in \overline{A \cup B}$. Then $x \notin A \cup B$, that is $x \notin A$ and $x \notin B$. Hence, $x \in \overline{A}$ and $x \in \overline{B}$. It follows that $x \in \overline{A} \cap \overline{B}$. Then $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$.

(ii) Now, let $x \in \overline{A} \cap \overline{B}$. Then $x \in \overline{A}$ and $x \in \overline{B}$. Hence, $x \notin A$ and $x \notin B$, that is $x \notin A \cup B$. Then $x \in \overline{A \cup B}$. Thus, $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$. By (i), it follows that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

□

Example 26. Let x be a real number. Show that $-x \leqslant |x|$ and $x \leqslant |x|$.

Proof. (i) Suppose x is a real number such that $x \geqslant 0$. Then $-x \leqslant 0 \leqslant |x|$. Further, we can write $|x| = x$. It follows that $x \leqslant |x|$. Hence, the statement is true for non-negative real numbers x.

(ii) Now, let x be a real number such that $x < 0$. Then $|x| = -x >\geqslant -x$. Further, we have that $x < 0 \leqslant |x|$. Hence, the statement is true for negative real numbers x. By (i), it follows that the statement is true for all real numbers x.

□

3.3 Proof by Contradiction

Recall from Section 2.2 on negation that another way of proving a true statement to be true is to show that the negation of the statement (now a false statement) is false. This is what we would call a proof by contradiction. Proofs by contradiction are not the most obvious to spot, but this would be the first choice whenever we cannot use a direct proof.

When we use a proof by contradiction, then it is important to first state the initial assumption and write out what the statement being false means. At the end of the proof, clearly announce that a contradiction has been found and state your conclusion.

Example 27. Show that 200 cannot be written as the sum of an odd integer and two even integers.

Proof. To the contrary, suppose that 200 can be written as the sum of an odd integer a and two even integers b and c. Since a is odd, there exists an integer n such that $a = 2n + 1$, and since b and c are even, there exist integers m and l such that $b = 2m$ and $c = 2l$. Then

$$200 = a + b + c = (2n + 1) + 2m + 2l = 2(n + m + l) + 1.$$

Since $n + m + l$ is an integer, we have that 200 is odd, which is a contradiction. The result follows. □

Example 28. Prove that if x and y are positive real numbers, then $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$.

Proof. To the contrary, suppose that there exist positive real numbers x and y such that $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$. Squaring both sides of the equation, we obtain

$$\begin{aligned} (\sqrt{x+y})^2 &= (\sqrt{x} + \sqrt{y})^2 \\ x + y &= x + 2\sqrt{x}\sqrt{y} + y \\ 0 &= 2\sqrt{x}\sqrt{y} \\ 0 &= \sqrt{xy}. \end{aligned}$$

But this implies that $xy = 0$. Then either $x = 0$ or $y = 0$ which is a contradiction since both x and y are positive. The result follows. □

4 Proof Techniques II

4.1 Proof by Contrapositive

A proof by contrapositive is using the fact that the statement $A \Rightarrow B$ is equivalent to its contrapositive $(\neg B) \Rightarrow (\neg A)$ (see Section 2.3 on contrapositives). This is a method which is used by choice. The question one may ask is, is it easier to prove $A \Rightarrow B$, or is it easier to prove $(\neg B) \Rightarrow (\neg A)$? If one prefers to prove the statement $(\neg B) \Rightarrow (\neg A)$, then one first assumes $\neg B$ and then proceeds to show that this implies $\neg A$. We call this a proof by contrapositive. This method should only be used if it simplifies the proof.

Example 29. Let a, b and c be integers, where a is non-zero. Prove that if $a \nmid bc$, then $a \nmid b$ and $a \nmid c$.

Proof. Suppose that a, b and c are integers such that $a \mid b$ or $a \mid c$. Without loss of generality, suppose that $a \mid b$. Then there exists an integer n such that $b = an$. Then

$$bc = (an)c = a(nc).$$

Since nc is an integer, we have that $a \mid bc$. The result follows by contrapositive. □

Remark 30. When giving a proof by contrapositive, we can either state that we are going to give a proof by contrapositive at the beginning of the proof, or we can state that the result follows *by contrapositive* at the end of the proof.

We will now discuss a special type of proof called an 'if and only if' proof. If $A \Leftrightarrow B$ is to be shown a true statement, then both $A \Rightarrow B$ and $B \Rightarrow A$ must be shown to be true statements. This follows since $A \Leftrightarrow B$ is equivalent to $(A \Rightarrow B) \wedge (B \Rightarrow A)$.

Example 31. Show that $5x - 11$ is even if and only if x is odd. This statement can be broken down into the following two substatements:

(i) If $5x - 11$ is even, then x is odd.

(ii) If x is odd, then $5x - 11$ is even.

Statement (ii) appears to be much more straight forward to prove than Statement (i). However, let us consider the contrapositive of (i) instead:

(i)(C) If x is even, then $5x - 11$ is odd.

Therefore, we will use a combination of a direct proof and a proof by contrapositive.

Proof. First, we will show that if $5x - 11$ is even, then x is odd by giving a proof by contrapositive. So suppose x is even. Then there exists an integer n such that $x = 2n$. Hence, we have that

$$5x - 11 = 5(2n) - 11 = 10n - 11 = (10n - 12) + 1 = 2(5n - 6) + 1.$$

Since $5n - 6$ is an integer, it follows that $5x - 11$ is odd as required.

Next, we will show that if x is odd, then $5x - 11$ is even. Since x is odd, then there exists an integer n such that $x = 2n + 1$. Hence, we have that

$$5x - 11 = 5(2n + 1) + 11 = 10n + 16 = 2(5n + 8).$$

Since $5n + 8$ is an integer, it follows that $5x - 11$ is even. The result follows. □

4.2 Proof by Induction

Induction is applied when we have an infinite number of statements which are indexed by the natural numbers as, for example, with the following statement:

$$2n + 6 \textit{ is even for all } n \in \mathbb{N}.$$

Here, it would not be sufficient to prove the statement for a sample of natural numbers, no matter how large the sample is. We have to prove it for *all* natural numbers! Hence, we first show that the statement is true for the smallest natural number. Then we assume a statement is true for some arbitrary natural number k and we proceed to show that the statement is true for its consecutive number, given by $k + 1$. That is, the truth of one statement implies the truth of the next statement. Since k was arbitrary and the statement is true for the smallest natural number and any two consecutive numbers, the statement is then true for all numbers. This is based on the Principle of Mathematical Induction:

Theorem 32 (The Principle of Mathematical Induction)**.** *For each positive integer n, let $P(n)$ be a statement. If*

(i) $P(1)$ is true, and

(ii) the implication

$$\textit{If } P(k), \textit{ then } P(k+1)$$

is true for every positive integer k,

then $P(n)$ is true for every positive integer n.

Example 33. Show that for every positive integer n, we have that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Proof. Let $P(n)$ be the following statement:

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

We will show that $P(n)$ is a true statement, for all $n \in \mathbb{N}$ by giving a proof by induction. First, let us consider $P(1)$. We have that

$$1^2 = \frac{1(2)(3)}{6} = 1.$$

Hence, $P(1)$ is a true statement.

Now, suppose that $P(k)$ is true for some natural number k, that is

$$1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

Then

$$\begin{aligned}
1^2 + 2^2 + \cdots + (k+1)^2 &= \left(1^2 + 2^2 + \cdots + k^2\right) + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)(k+1)}{6} \\
&= \frac{k(k+1)(2k+1) + 6(k+1)(k+1)}{6} \\
&= \frac{(k+1)(k(2k+1) + 6(k+1))}{6} \\
&= \frac{(k+1)(2k^2 + k + 6k + 6)}{6} \\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6},
\end{aligned}$$

showing that if $P(k)$ is true then $P(k+1)$ is true. By the Principle of Mathematical Induction, it follows that $P(n)$ is true for all natural numbers n. □

The method for writing a proof by induction is the following:

(i) We announce that we are going to give a proof by induction.

(ii) We prove the initial case (the base case).

(iii) We state that we are assuming that the statement is true for some natural number k and write out what this means.

(iv) We use the truth of the statement for k in the proof of the statement for $k+1$. Often this involves breaking down the equation (or inequality) into two pieces- one of which involves the case for k.

(v) We state the conclusion : *By the Principle of Mathematical Induction the statement is true for all natural numbers.*

Here, it is important to note that we do not necessarily start with $n = 1$ as our initial case. As a matter of fact, this may not be true for some statements. Consider the following example:

Example 34. Show that $n^2 \leqslant 2^{n-1}$ for all natural numbers n such that $n \geqslant 7$.

In the above example, we have that the statement is certainly true for $n = 1$. However, it is **not** true for $n = 2, 3, 4, 5$, and 6. If we encounter an example like the above, we need to take the following steps in order to prove that a statement $P(n)$ is true for all integers n such that $n \geqslant r$, for some natural number r:

(i) We show that $P(r)$ is true.

(ii) We show that if $P(k)$ is true for some natural number $k \geqslant r$, then $P(k+1)$ is true.

This would prove that $P(n)$ is true for all $n \geqslant r$ by the Principle of Mathematical Induction.

Example 35 (Proof of Example 34)**.** Let $P(n)$ be the following statement:

$$n^2 \leqslant 2^{n-1}.$$

We will show that $P(n)$ is a true statement, for all natural numbers n such that $n \geqslant 7$, by giving a proof by induction. First, let us consider $n = 7$. We have that $7^2 = 49 \leqslant 64 = 2^6$. Hence, $P(7)$ is a true statement.

Now, suppose that $P(k)$ is a true statement for some natural number $k \geqslant 7$, that is

$$k^2 \leqslant 2^{k-1}.$$

Then

$$\begin{aligned}
(k+1)^2 &= k^2 + 2k + 1 \\
&\leqslant k^2 + 2k + k, && \text{since } k \geqslant 7, \\
&= k^2 + 3 \cdot k \\
&\leqslant k^2 + k \cdot k, && \text{since } k \geqslant 7, \\
&= 2k^2 \\
&\leqslant 2 \cdot 2^{k-1}, && \text{by assumption,} \\
&= 2^k \\
&= 2^{(k+1)-1},
\end{aligned}$$

showing that if $P(k)$ is true then $P(k+1)$ is true. By the Principle of Mathematical Induction, it follows that $P(n)$ is true for all natural numbers $n \geqslant 7$.

4.2.1 The Strong Principle of Mathematical Induction

Consider the following example:

Example 36. Suppose that $x_1 = 3$ and $x_2 = 5$ and that, for $n \geqslant 3$, we have that

$$x_n = 3x_{n-1} - 2x_{n-2}.$$

Show that $x_n = 2^n + 1$, for all $n \in \mathbb{N}$.

Here, it is not sufficient to assume the statement is true for some natural number k. We must also have that the statement is true for $k-1$ and $k-2$ (since $x_k = 3x_{k-1} - 2x_{k-1}$). Therefore, we would assume that the statement is true for all integers i such that $3 \leqslant i \leqslant k$. This would be a case when we use the Strong Principle of Mathematical Induction:

Theorem 37 (The Strong Principle of Mathematical Induction)**.** *Let m be a fixed integer, and for each positive integer $n \geqslant m$, let $P(n)$ be a statement. If*

(i) $P(m)$ is true, and

(ii) the implication

$$\textit{If } P(i), \textit{ for } m \leqslant i \leqslant k, \textit{ then } P(k+1)$$

is true for every positive integer k,

then $P(n)$ is true for every positive integer n.

Example 38 (Proof of Example 36)**.** *Proof.* Let $P(n)$ be the following statement:

$$x_n = 2^n + 1.$$

We will show that $P(n)$ is a true statement, for all $n \in \mathbb{N}$, by giving a proof by induction. First, let us consider $n = 1$ and $n = 2$. We have the given initial conditions $x_1 = 3$ and $x_2 = 5$. Using the formula $x_n = 2^n + 1$, we have

$$x_1 = 2^1 + 1 = 3 \qquad \text{and} \qquad x_2 = 2^2 + 1 = 5,$$

as required. Therefore, $P(1)$ and $P(2)$ are true statements.

Now, suppose $P(i)$ is a true statement, for all $2 \leqslant i \leqslant k$. Hence, we have that x_{k-1} and x_k are true statements. That is, we have

$$x_{k-1} = 2^{k-1} + 1$$
$$x_k = 2^k + 1.$$

Then

$$\begin{aligned} x_{k+1} &= 3x_k - 2x_{k-1} \\ &= 3(2^k + 1) - 2(2^{k-1} + 1) \\ &= 3 \cdot 2^k - 2 \cdot 2^{k-1} + 1 \\ &= 3 \cdot 2^k - 2^k + 1 \\ &= 2 \cdot 2^k + 1 \\ &= 2^{k+1} + 1, \end{aligned}$$

showing that if $P(i)$ is true, for $2 \leqslant i \leqslant k$ then $P(k+1)$ is true. By the Strong Principle of Mathematical Induction, it follows that $P(n)$ is true for all natural numbers n. □

Note that for our initial case, we showed both $n = 1$ and $n = 2$ to be true. Why? Well, we are claiming that $x_n = 2n+1$, for all natural numbers n and we are given that $x_1 = 3$ and $x_2 = 5$. Of course we must first check that our given values satisfy the formula! Also, for our inductive hypothesis we assume that $P(i)$ is true for all $2 \leqslant i \leqslant k$. We chose $i \geqslant 2$ rather than $i \geqslant 1$ because we already showed $P(1)$ and $P(2)$ to be true. Further, using the formula for x_{k+1}, we must have that x_k and x_{k-1} are true. But this only makes sense if $k - 1 \geqslant 1$, that is $k \geqslant 2$.

4.3 Disproof by Counter Example

Often, we come across statements involving *for all*. In order to prove such a statement, we must prove that the statement is true in general. For example, a proof by induction can be used to show that a statement is true for all natural numbers n. But what happens when the given statement is false?

Recall that the negation of a true statement is a false statement. Therefore, the negation of a false statement is a true statement. Hence, if we want to disprove a statement, that is show that the statement is false, we can prove that the negation of the statement is true. The negation of *for all* is *there exists*. Therefore, in order to prove that the negation of a statement involving *for all* is true, we must give an example that something exists- one example is sufficient! We call such an example a **counterexample**.

Example 39. Show that all primes are odd.

Proof. We will disprove the statement by giving a counterexample. We have that 2 is prime and is even. Therefore, the above statement is false. □

Remark 40. Here, it is important to announce that you are going to 'disprove' the statement by giving a counterexample rather than actually proving the statement to be true.

Another type of statement which can be disproved by counterexample would be a statement of the form $A \Rightarrow B$. In order to show that $A \Rightarrow B$ is a true statement, we must show that this is true in general. Hence, to disprove the statement, it is sufficient to find one example, a counterexample, where the statement is false.

Example 41. Show that if x is real number, then $2x$ is even.

Proof. We will disprove the statement by giving a counterexample. We let $x = \frac{1}{2}$. Then x is a real number, and $2x = 2\left(\frac{1}{2}\right) = 1$ which is odd. Therefore, the above statement is false. □