Ramanujan-style congruences of local origin

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Ramanujan's congruence

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$$= q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 + \cdots$$

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So, we have

n	1	2	3	4	5	6
$\tau(n)$	1	-24	252	-1472	4830	-6048

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Ramanujan's discovery

$\tau(n) \equiv \sigma_{11}(n) \bmod 691$

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Proof (Sketch)

Consider the normalised, weight 12 Eisenstein series,

$$E_{12}(z) = \frac{B_{12}}{24} + \sum_{n \geq 1} \sigma_{11}(n) q^n \in M_{12}(SL_2(\mathbb{Z}))$$

and note that the discriminant function is a weight 12 cusp form,

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- Where does 691 appear? 691 divides B_{12}, so the constant coefficient of E_{12} vanishes modulo 691. E_{12} and Δ are both modular forms of weight 12 whose constant terms vanish modulo 691 - "cusp forms mod 691".
- Comparing the other coefficients we conclude that E_{12} and Δ must be the same cusp form mod 691.

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In general, the existence of Eisenstein congruences like Ramanujan's leads to the proof of the Herbrand-Ribet Theorem, a major theorem in number theory concerning class groups of cyclotomic fields and the associated Galois action.

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Q2: How do we find them, e.g. modulo what primes?

Q3: What can we say about how "new" the modular forms that satisfy the congruence are?

Brief definitions

A **modular form** is a holomorphic function on the upper half plane which satifies a transformation formula under the action of $SL_2(\mathbb{Z})$.

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A modular form of **level** N only has to satisfy the transformation formula for a subgroup $\Gamma_1(N) \subset SL_2(\mathbb{Z})$ which depends on N.

A modular form is **new** at level N if it is not a lift of a modular form from some lower level, i.e. it first appears at level N.

A general conjecture

Conjecture (Fretwell, R.)

Assume we have

- N, M - coprime, squarefree integers, $k > 2$ - integer.
- $E_k^{\psi,\phi}$ - new at level N with $\psi\phi = \chi$.
- $l > k+1$, $l \nmid 6NM$ - prime of $\mathbb{Z}[\psi, \phi]$.
- λ - prime above l in the ring of integers of the extension of $\mathbb{Q}(\psi, \phi)$ generated by the Fourier coefficients of f.

Then for all primes $p \nmid NMl$, there exists a newform $f \in S_k(\Gamma_1(NM), \tilde{\chi})$ such that

$$a_p(f) \equiv a_p(E_k^{\psi,\phi}) \pmod{\lambda}$$

if and only if both of the following hold:

1. $\mathrm{ord}_l(L(1-k, \psi^{-1}\phi) \prod_{p \in \mathcal{P}_M} (\psi(p) - \phi(p)p^k)) > 0$.
2. $l \mid (\psi(p) - \phi(p)p^k)(\psi(p) - \phi(p)p^{k-2}) \ \forall\ p \in \mathcal{P}_M$.

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- We can find a congruence between a "new" cuspform $f = \sum_{n \geq 1} a_n q^n$ and an Eisenstein series $E_k^{\psi,\phi}$.
- The congruence held modulo 691, this worked because 691 divided B_{12} (the constant coefficient of E_{12}).
- The congruence holds modulo prime l, where l divides either the constant term of the Eisenstein series or an Euler product (some other information depending on the "level" of f).
- To ensure f is "new", l also has to divide other information depending on the "level" of f.

Consequences/ future work

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- Our conjecture should provide evidence for a case of the Bloch-Kato conjecture - a deep conjecture in algebraic number theory relating L-values and arithmetic of Galois modules.
- In particular, it should tell us that a certain Bloch-Kato Selmer group (a wide generalisation of class groups) is non-trivial.
- We are also working on generalising our results to Hilbert modular forms.

Thank you!