Matrix superalgebras and lattice isometry

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What is a superalgebra?

Let A and B be vector subspaces of some algebra X, we say they form a superalgebra over X if:

- $A \oplus B = X$.
- $AA \subset A, BB \subset A, AB \subset B, BA \subset B$.
  - A is the 'even' component, B is the 'odd' component.

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Our goal for the next few slides is to give a concrete example of a superalgebra over $M_n(K)$, for K a field with $\text{char}(K) = 0$, first constructed by Hill, Lettington & Schmidt in 2017.

Defining S_n

Let a matrix M be in $S_n \subset M_n(K)$ if

$$\sum_{i=1}^{n} M_{ij} = \sum_{i=1}^{n} M_{ji} = w \text{ for all } 1 \leq j \leq n.$$

Equivalently, taking $1_n = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{Z}^n$, we can write this as

$$1_n^T M = w 1_n^T, \; M 1_n = w 1_n.$$

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Now, taking $\langle \cdot, \cdot \rangle$ to be the standard bilinear form $\langle a, b \rangle = a^T b$ for $a, b \in K^n$, we also have that:

$$M \in S_n \iff \langle u, M1_n \rangle = 0,\ \langle 1_n, Mu \rangle = 0\ \forall\ u \in \{1_n\}^{\perp}.$$

Properties of S_n

$S_n = \{M \in M_n(K) : \langle u, M1_n \rangle = 0,\ \langle 1_n, Mu \rangle = 0 \ \forall\ u \in \{1_n\}^{\perp}\}$.

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- Define the weight of a matrix M to be

$$\mathsf{wt}(M) = \frac{1}{n^2} \sum_{i,j=1}^{n} M_{ij} = \frac{1}{n^2} 1_n^T M 1_n.$$

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- Then $M \in S_n$ if and only if we can write:

$$M = M_0 + \mathrm{wt}(M)\varepsilon_n$$

where M_0 is such that $\mathrm{wt}(M_0) = 0$ and ε_n is the $n \times n$ all 1s matrix ($\varepsilon_n = 1_n 1_n^T$).

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- $\dim(S_n) = n^2 - 2(n-1) = n^2 - 2n + 2$.

S_2 example

Consider $S_2 \subset M_2(K)$, then $S_2 = \text{span}\left(\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right), \left(\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right)\right)$.
Then, $\text{wt}\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right) = \frac{1}{2}$ and we can write:

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \underbrace{\frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}}_{M_0} + \underbrace{\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}}_{\text{wt}(M)\varepsilon_n}$$

Defining V_n

Let a matrix M be in $V_n \subset M_n(K)$ if

$$M_{ij} + M_{kl} = M_{il} + M_{kj},\ \ i,j,k,l \in \{1, \cdots, n\} \text{ and } \sum_{i,j=1}^{n} M_{ij} = 0$$

$$\begin{pmatrix} \color{red}{*} & * & \color{blue}{*} \\ * & * & * \\ \color{blue}{*} & * & \color{red}{*} \end{pmatrix} \begin{pmatrix} \color{red}{*} & * & \color{blue}{*} \\ \color{blue}{*} & * & \color{red}{*} \\ * & * & * \end{pmatrix} \begin{pmatrix} \color{red}{*} & \color{blue}{*} & * \\ * & * & * \\ \color{blue}{*} & \color{red}{*} & * \end{pmatrix}$$

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Using the bilinear form $\langle \cdot, \cdot \rangle$, we have:

$$M \in V_n \iff \langle u, Mv \rangle = 0\ \forall\ u, v \in \{1_n\}^{\perp},\ \langle 1_n, M1_n \rangle = 0.$$

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$V_n = \{M \in M_n(K) : \langle u, Mv \rangle = 0 \ \forall \ u, v \in \{1_n\}^{\perp}, \ \langle 1_n, M1_n \rangle = 0\}$.

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- $\dim(V_n) = 2n - 2$.

V_2 example

Consider $V_2 \subset M_2(K)$, then $V_2 = \text{span}\left(\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right), \left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)\right)$. Then,

$$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \underbrace{\frac{1}{2}\begin{pmatrix}1\\ -1\end{pmatrix}\begin{pmatrix}1 & 1\end{pmatrix}}_{a1_n^T} + \underbrace{\frac{1}{2}\begin{pmatrix}1\\ 1\end{pmatrix}\begin{pmatrix}1 & -1\end{pmatrix}}_{1_n b^T}$$

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Recap

S_n	V_n
$u^T M 1_n = 1_n^T M u = 0 \;\; \forall\, u \in \{1_n\}^\perp$	$u^T M v = 1_n^T M 1_n = 0 \;\; \forall\, u, v \in \{1_n\}^\perp$
$M_0 + \mathrm{wt}(M)\varepsilon_n$	$a1_n^T + 1_n b^T$ for some $a, b \in \{1_n\}^\perp$
$\dim(S_n) = n^2 - 2n + 2$	$\dim(V_n) = 2n - 2$

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Assume $M \in S_n \cap V_n$ then, for all $u, v \in \{1_n\}^{\perp}$, M satisfies:

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So, for all $u \in \{1_n\}^\perp$, Mu must be orthogonal to all of K^n, hence $Mu = 0$.

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Since span$(1_n, \{1_n\}^\perp) = K^n$, we know $Ma = 0$ for all $a \in K^n$, hence $M = \mathbf{0}_n$.

Hence, $S_n \oplus V_n = M_n(K)$.

Superalgebra property

To check we have a superalgebra, we need:

$$S_n S_n \subset S_n, \; V_n V_n \subset S_n, \; S_n V_n \subset V_n, \; V_n S_n \subset V_n.$$

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$$S_nS_n \subset S_n,\ V_nV_n \subset S_n,\ S_nV_n \subset V_n,\ V_nS_n \subset V_n.$$

Let $S_1, S_2 \in S_n$, then

$$\langle 1_n, S_1S_2u\rangle = 1_n^T S_1S_2u = w_1 1_n^T S_2u = 0.$$

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$$\langle u, S_1 S_2 1_n \rangle = u^T S_1 S_2 1_n = w_2 u^T S_1 1_n = 0.$$

$\implies S_1 S_2 \in S_n.$

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$\implies S_1S_2 \in S_n$. Others follow similarly, so S_n and V_n form a superalgebra.

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Let $S_1, S_2 \in S_n$, then

$$\langle 1_n, S_1S_2u\rangle = 1_n^TS_1S_2u = w_1 1_n^TS_2u = 0.$$

$$\langle u, S_1S_2 1_n\rangle = u^TS_1S_2 1_n = w_2u^TS_1 1_n = 0.$$

$\implies S_1S_2 \in S_n$. Others follow similarly, so S_n and V_n form a superalgebra.

For any matrix $M \in M_n(K)$, we can decompose M as $M = S + V$ for some $S \in S_n$ and $V \in V_n$. In particular,

$$M = M_0 + \text{wt}(M)\varepsilon_n + a1_n^T + 1_nb^T$$

Lattice isometry problem

- Take two rank n $\mathbb{Z}$-lattices Λ_M and Λ_B with associated Gram matrices $M, B \in GL_n(\mathbb{Z})$.
- If Λ_M and Λ_B are isometric, then we can write $M = N^T B N$ for some $N \in GL_n(\mathbb{Z})$.
- We want to use the superalgebra structure on the above equation to try to determine if its possible for the lattices to be isometric. In the case $B = I_n$, Higham, Lettington & Schmidt (2021) studied this problem to see when $M = N^T N$.

Lattice isometry problem

- Take two rank n $\mathbb{Z}$-lattices Λ_M and Λ_B with associated Gram matrices $M, B \in GL_n(\mathbb{Z})$.
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- We want to use the superalgebra structure on the above equation to try to determine if its possible for the lattices to be isometric. In the case $B = I_n$, Higham, Lettington & Schmidt (2021) studied this problem to see when $M = N^T N$.

Idea: Decompose M into S_n and V_n parts and compare with the S_n and V_n parts of $N^T BN$.

To do this, we need to take $K = \mathbb{Q}$ and generalise the ideas of S_n and V_n.

The new superalgebra

Take B to be a symmetric, positive definite matrix in $GL_n(\mathbb{Z})$. Define $\langle \cdot, \cdot \rangle_B$ to be the vector inner product $\langle a, b \rangle_B = a^T Bb$. Then

$$S_{n,B} = \{M \in M_n(\mathbb{Q}) : \langle u, M1_n \rangle_B = 0, \ \langle 1_n, Mu \rangle_B = 0 \ \forall \ u \in \{1_n\}_B^{\perp}\}.$$
$$V_{n,B} = \{M \in M_n(\mathbb{Q}) : \langle 1_n, M1_n \rangle_B = 0, \ \langle u, Mv \rangle_B = 0 \ \forall \ u, v \in \{1_n\}_B^{\perp}\}.$$

where $u \in \{1_n\}_B^{\perp} \iff \langle u, 1_n \rangle_B = u^T B 1_n = 0$.

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$$V_{n,B} = \{M \in M_n(\mathbb{Q}) : \langle 1_n, M1_n\rangle_B = 0,\ \langle u, Mv\rangle_B = 0\ \forall\ u,v \in \{1_n\}^{\perp}_{B}\}.$$

where $u \in \{1_n\}^{\perp}_{B} \iff \langle u, 1_n\rangle_B = u^T B1_n = 0$.

Define weight w.r.t. B as

$$\mathrm{wt}(M)_B = \frac{1_n^T BM1_n}{1_n^T B1_n} = \frac{1_n^T BM1_n}{n^2\mathrm{wt}(B)}$$

The new superalgebra

Then

$$M \in S_{n,B} \iff M = M_0 + \frac{\mathsf{wt}(M)_B}{n^2\mathsf{wt}(B)}\varepsilon_n B, \text{ for some } M_0 \text{ s.t. } \mathsf{wt}(M_0)_B = 0.$$

$$M \in V_{n,B} \iff M = a1_n^T B + 1_n b^T B \text{ for some } a, b \in \{1_n\}_B^\perp.$$

It turns out $S_{n,B} \oplus V_{n,B} = M_n(\mathbb{Q})$ is also a superalgebra. So for any $M \in M_n(\mathbb{Q})$, we can write:

$$M = M_0 + \frac{\mathsf{wt}(M)_B}{n^2\mathsf{wt}(B)}\varepsilon_n B + a1_n^T B + 1_n b^T B.$$

Application to lattice isometry

Let's assume that M and B are the gram matrices of two isometric lattices. Then we can write $M = N^T BN$. Taking the weight of M (multiplied by n^2):

$$n^2\mathsf{wt}(M)$$

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&= 1_n^T N^T B(N_0 + \omega_N \varepsilon_n B + a 1_n^T B + 1_n b^T B) 1_n
\end{aligned}
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Application to lattice isometry

Let's assume that M and B are the gram matrices of two isometric lattices. Then we can write $M = N^T BN$. Taking the weight of M (multiplied by n^2):

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$$\omega_N = \frac{\mathsf{wt}(N)_B}{n^2\mathsf{wt}(B)} = \frac{1_n^T B N 1_n}{n^4\mathsf{wt}(B)^2} \implies 1_n^T N^T B 1_n = 1_n^T B N 1_n = n^4\mathsf{wt}(B)^2\omega_N$$

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So, if Λ_M and Λ_B are rank n, isometric lattices, we must have an integer solution to the above quadratic form.

Rank 2 example

Take $B = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 5 \end{smallmatrix}\right)$ and $M = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 3 \end{smallmatrix}\right)$.

$n^2 \mathsf{wt}(B) = 6, \mathsf{wt}(M) = \frac{7}{4}$ and $a^T B a = a_1^2 + 5a_2^2$.

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Using the condition that $1_n^T Ba = a_1 + 5a_2 = 0$, we have $a_1 = -5a_2$. Taking $\tilde{\omega} = 36\omega_M \in \mathbb{Z}, \tilde{a} = 36a_2 \in \mathbb{Z}$, we have

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So $|\tilde{a}| \leq 2$, and after trying all options, it is clear there are no integer solutions, so B and M cannot be isometric!

Thank you!