

Newform Eisenstein congruences of local origin

Jenny Roberts
(joint work with Dan Fretwell)

University of Bristol

7 September 2023

Ramanujan's congruence

Recall: the discriminant function $\Delta \in S_{12}(SL_2(\mathbb{Z}))$ has q -expansion

$$\Delta(z) = q \prod_{n \geq 1} (1 - q^n)^{24}$$

Ramanujan's congruence

Recall: the discriminant function $\Delta \in S_{12}(SL_2(\mathbb{Z}))$ has q -expansion

$$\begin{aligned}\Delta(z) &= q \prod_{n \geq 1} (1 - q^n)^{24} = \sum_{n \geq 1} \tau(n) q^n \\ &= q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 + \dots\end{aligned}$$

Ramanujan's congruence

Recall: the discriminant function $\Delta \in S_{12}(SL_2(\mathbb{Z}))$ has q -expansion

$$\begin{aligned}\Delta(z) &= q \prod_{n \geq 1} (1 - q^n)^{24} = \sum_{n \geq 1} \tau(n) q^n \\ &= q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 + \dots\end{aligned}$$

and the Eisenstein series $E_{12} \in M_{12}(SL_2(\mathbb{Z}))$ has q -expansion

$$\begin{aligned}E_{12}(z) &= -\frac{B_{12}}{24} + \sum_{n \geq 1} \sigma_{11}(n) q^n \in M_{12}(SL_2(\mathbb{Z})) \\ &= \frac{691}{65520} + q + 2049q^2 + 177148q^3 + 4196353q^4 \\ &\quad + 48828126q^5 + 362976252q^6 + \dots\end{aligned}$$

Ramanujan's congruence

If we were to compare the Fourier coefficients of these two series...

n	1	2	3	4	5	6
$\tau(n)$	1	-24	252	-1472	4830	-6048
$\sigma_{11}(n)$	1	2049	177148	4196353	48828126	362976252

Ramanujan's congruence

If we were to compare the Fourier coefficients of these two series...

	n	1	2	3	4	5	6
$\tau(n)$	$\text{mod } 691$	1	667	252	601	684	171
$\sigma_{11}(n)$	$\text{mod } 691$	1	667	252	601	684	171

Ramanujan's congruence

If we were to compare the Fourier coefficients of these two series...

n	1	2	3	4	5	6
$\tau(n) \pmod{691}$	1	667	252	601	684	171
$\sigma_{11}(n) \pmod{691}$	1	667	252	601	684	171

Ramanujan's congruence

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}$$

Ramanujan's congruence

Why does this congruence hold?

Ramanujan's congruence

Why does this congruence hold?

Proof (Sketch)

We have

$$E_{12}(z) = -\frac{B_{12}}{24} + \sum_{n \geq 1} \sigma_{11}(n)q^n \in M_{12}(SL_2(\mathbb{Z}))$$

and

$$\Delta(z) = \sum_{n \geq 1} \tau(n)q^n \in S_{12}(SL_2(\mathbb{Z})).$$

- Since 691 divides the numerator of B_{12} we see that E_{12} reduces to an eigenform $\overline{E}_{12} \in S_{12}(SL_2(\mathbb{Z}), \overline{\mathbb{F}}_{691})$ - a cusp form 'mod 691'.

Ramanujan's congruence

Why does this congruence hold?

Proof (Sketch)

We have

$$E_{12}(z) = -\frac{B_{12}}{24} + \sum_{n \geq 1} \sigma_{11}(n)q^n \in M_{12}(SL_2(\mathbb{Z}))$$

and

$$\Delta(z) = \sum_{n \geq 1} \tau(n)q^n \in S_{12}(SL_2(\mathbb{Z})).$$

- Since 691 divides the numerator of B_{12} we see that E_{12} reduces to an eigenform $\overline{E}_{12} \in S_{12}(SL_2(\mathbb{Z}), \overline{\mathbb{F}}_{691})$ - a cusp form 'mod 691'.
- This space is spanned by the reductions of eigenforms in $S_{12}(SL_2(\mathbb{Z}), \mathbb{Z}_{691})$, which is 1-dimensional spanned by Δ .

Ramanujan's congruence

Why does this congruence hold?

Proof (Sketch)

We have

$$E_{12}(z) = -\frac{B_{12}}{24} + \sum_{n \geq 1} \sigma_{11}(n)q^n \in M_{12}(SL_2(\mathbb{Z}))$$

and

$$\Delta(z) = \sum_{n \geq 1} \tau(n)q^n \in S_{12}(SL_2(\mathbb{Z})).$$

- Since 691 divides the numerator of B_{12} we see that E_{12} reduces to an eigenform $\overline{E}_{12} \in S_{12}(SL_2(\mathbb{Z}), \overline{\mathbb{F}}_{691})$ - a cusp form 'mod 691'.
- This space is spanned by the reductions of eigenforms in $S_{12}(SL_2(\mathbb{Z}), \mathbb{Z}_{691})$, which is 1-dimensional spanned by Δ .
- Then $\overline{E}_{12} = \overline{\Delta}$, implying the congruence.

Why do we care?

It turns out, this congruence has deep consequences...

Why do we care?

It turns out, this congruence has deep consequences...

There exists a non trivial $[\mathfrak{a}] \in \text{Cl}(\mathbb{Q}(\zeta_{691}))[691]$ satisfying:

$$\sigma \cdot [\mathfrak{a}] = \chi_{691}(\sigma)^{-11} [\mathfrak{a}] \text{ for all } \sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}),$$

where $\chi_{691} : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathbb{F}_{691}^*$ is the mod 691 cyclotomic character.

Why do we care?

It turns out, this congruence has deep consequences...

There exists a non trivial $[\mathfrak{a}] \in \text{Cl}(\mathbb{Q}(\zeta_{691}))[691]$ satisfying:

$$\sigma \cdot [\mathfrak{a}] = \chi_{691}(\sigma)^{-11} [\mathfrak{a}] \text{ for all } \sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}),$$

where $\chi_{691} : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathbb{F}_{691}^*$ is the mod 691 cyclotomic character.

More generally, Eisenstein congruences were used to prove:

Herbrand-Ribet Theorem

For $2 \leq k \leq p-3$ even:

$$\text{ord}_p(B_k) > 0 \iff \exists \text{ element in the } \chi_p^{1-k} \text{ eigenspace of } \text{Cl}(\mathbb{Q}(\zeta_p))[p].$$

Next steps

Q1: Can we find other congruences like this, but for higher levels and non-trivial character?

Next steps

Q1: Can we find other congruences like this, but for higher levels and non-trivial character?

- Higher levels: Dummigan-Fretwell (2014), Billerey-Menares (2016)
- Non-trivial character: Minimal level - Dummigan (2007), lift by prime level - Spencer (2018)

Next steps

Q1: Can we find other congruences like this, but for higher levels and non-trivial character?

- Higher levels: Dummigan-Fretwell (2014), Billerey-Menares (2016)
- Non-trivial character: Minimal level - Dummigan (2007), lift by prime level - Spencer (2018)

Q2: What can we say about how “new” the modular forms that satisfy the congruence are?

Next steps

Q1: Can we find other congruences like this, but for higher levels and non-trivial character?

- Higher levels: Dummigan-Fretwell (2014), Billerey-Menares (2016)
- Non-trivial character: Minimal level - Dummigan (2007), lift by prime level - Spencer (2018)

Q2: What can we say about how “new” the modular forms that satisfy the congruence are?

- Trivial character: Dummigan-Fretwell (2014), Billerey-Menares (2016)

Next steps

Q1: Can we find other congruences like this, but for higher levels and non-trivial character?

- Higher levels: Dummigan-Fretwell (2014), Billerey-Menares (2016)
- Non-trivial character: Minimal level - Dummigan (2007), lift by prime level - Spencer (2018)

Q2: What can we say about how “new” the modular forms that satisfy the congruence are?

- Trivial character: Dummigan-Fretwell (2014), Billerey-Menares (2016)

Q3: How do we find them, e.g. modulo what primes?

- Generalise Ribet’s converse to Herbrand’s theorem

Next steps

Q1: Can we find other congruences like this, but for higher levels and non-trivial character?

- Higher levels: Dummigan-Fretwell (2014), Billerey-Menares (2016)
- Non-trivial character: Minimal level - Dummigan (2007), lift by prime level - Spencer (2018)

Q2: What can we say about how “new” the modular forms that satisfy the congruence are?

- Trivial character: Dummigan-Fretwell (2014), Billerey-Menares (2016)

Q3: How do we find them, e.g. modulo what primes?

- Generalise Ribet’s converse to Herbrand’s theorem

Goal: Generalise to newform congruences with non-trivial character and lift by squarefree level.

Definitions - Congruence subgroups

Define the congruence subgroups of $SL_2(\mathbb{Z})$:

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

and

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\}.$$

Definitions - Modular forms of level N

Let $M_k(\Gamma_0(N), \chi)$ be the space of modular forms of weight $k \geq 2$, level N and Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ with modulus N .

Definitions - Modular forms of level N

Let $M_k(\Gamma_0(N), \chi)$ be the space of modular forms of weight $k \geq 2$, level N and Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ with modulus N .

Definition (Modular form)

$f \in M_k(\Gamma_0(N), \chi)$ if:

- f is holomorphic on the complex upper half plane
- f satisfies:

$$f[\gamma]_k := (cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right) = \chi(d) f(z)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

- $f[\alpha]_k$ must be holomorphic at ∞ for all $\alpha \in SL_2(\mathbb{Z})$, i.e. at all cusps.

Definitions - Generalised Eisenstein series

We can decompose $M_k(\Gamma_0(N), \chi)$ as

$$M_k(\Gamma_0(N), \chi) = S_k(\Gamma_0(N), \chi) \oplus E_k(\Gamma_0(N), \chi)$$

If $k > 2$, the Eisenstein subspace $E_k(\Gamma_0(N), \chi)$ is spanned by the normalised Eisenstein series $E_k^{\psi, \phi}(tz)$ for all ordered pairs of Dirichlet characters ϕ, ψ of conductors u, v satisfying $\psi\phi = \chi$ and $tuv \mid N$. $E_k^{\psi, \phi}$ has Fourier expansion:

$$E_k^{\psi, \phi}(z) = \frac{1}{2}\delta(\psi)L(1-k, \psi^{-1}\phi) + \sum_{n=1}^{\infty} \sigma_{k-1}^{\psi, \phi}(n)q^n,$$

$\delta(\psi) = 1$ if ψ is the trivial mod 1 character, 0 otherwise and

$$\sigma_{k-1}^{\psi, \phi}(n) = \sum_{d|n, d>0} \psi(n/d)\phi(d)d^{k-1}$$

is the generalised power divisor series.

Definitions - Newforms

The cuspidal subspace has an orthogonal decomposition with respect to the Petersson inner product:

$$S_k(\Gamma_0(N), \chi) = S_k^{\text{old}}(\Gamma_0(N), \chi) \oplus S_k^{\text{new}}(\Gamma_0(N), \chi)$$

with $S_k^{\text{old}}(\Gamma_0(N), \chi)$ as usual being spanned by lifts of modular forms in $S_k(\Gamma_0(N/d), \chi)$ with $d \cdot \text{cond}(\chi) \mid N$.

- A modular form is **new** at level N if it lies in the new subspace.

Definitions - Newforms

The cuspidal subspace has an orthogonal decomposition with respect to the Petersson inner product:

$$S_k(\Gamma_0(N), \chi) = S_k^{\text{old}}(\Gamma_0(N), \chi) \oplus S_k^{\text{new}}(\Gamma_0(N), \chi)$$

with $S_k^{\text{old}}(\Gamma_0(N), \chi)$ as usual being spanned by lifts of modular forms in $S_k(\Gamma_0(N/d), \chi)$ with $d \cdot \text{cond}(\chi) \mid N$.

- A modular form is **new** at level N if it lies in the new subspace.

Definitions - Newforms

The cuspidal subspace has an orthogonal decomposition with respect to the Petersson inner product:

$$S_k(\Gamma_0(N), \chi) = S_k^{\text{old}}(\Gamma_0(N), \chi) \oplus S_k^{\text{new}}(\Gamma_0(N), \chi)$$

with $S_k^{\text{old}}(\Gamma_0(N), \chi)$ as usual being spanned by lifts of modular forms in $S_k(\Gamma_0(N/d), \chi)$ with $d \cdot \text{cond}(\chi) \mid N$.

- A modular form is **new** at level N if it lies in the new subspace.
- An **eigenform** is an eigenvector for all the Hecke operators, T_p , with $p \nmid N$.

Definitions - Newforms

The cuspidal subspace has an orthogonal decomposition with respect to the Petersson inner product:

$$S_k(\Gamma_0(N), \chi) = S_k^{\text{old}}(\Gamma_0(N), \chi) \oplus S_k^{\text{new}}(\Gamma_0(N), \chi)$$

with $S_k^{\text{old}}(\Gamma_0(N), \chi)$ as usual being spanned by lifts of modular forms in $S_k(\Gamma_0(N/d), \chi)$ with $d \cdot \text{cond}(\chi) \mid N$.

- A modular form is **new** at level N if it lies in the new subspace.
- An **eigenform** is an eigenvector for all the Hecke operators, T_p , with $p \nmid N$.
- $f = \sum_{n \geq 0} a_n(f)q^n \in M_k(\Gamma_0(N), \chi)$ is **normalised** if $a_1(f) = 1$.

Definitions - Newforms

The cuspidal subspace has an orthogonal decomposition with respect to the Petersson inner product:

$$S_k(\Gamma_0(N), \chi) = S_k^{\text{old}}(\Gamma_0(N), \chi) \oplus S_k^{\text{new}}(\Gamma_0(N), \chi)$$

with $S_k^{\text{old}}(\Gamma_0(N), \chi)$ as usual being spanned by lifts of modular forms in $S_k(\Gamma_0(N/d), \chi)$ with $d \cdot \text{cond}(\chi) \mid N$.

- A modular form is **new** at level N if it lies in the new subspace.
- An **eigenform** is an eigenvector for all the Hecke operators, T_p , with $p \nmid N$.
- $f = \sum_{n \geq 0} a_n(f)q^n \in M_k(\Gamma_0(N), \chi)$ is **normalised** if $a_1(f) = 1$.
- We say $f \in M_k(\Gamma_0(N), \chi)$ is a **newform** if f is a normalised eigenform which is new at level N .

A general conjecture

Assume we have

- N, M - coprime, squarefree integers, $k > 2$ - integer.
- $E_k^{\psi, \phi}$ - new at level N ($\text{cond}(\psi) \cdot \text{cond}(\phi) = N$) with $\psi\phi = \chi$, $\tilde{\chi}$ is a lift of χ to modulus NM .
- $l > k + 1$, $l \nmid NM$ - prime of $\mathbb{Z}[\psi, \phi]$.
- λ - prime above l in the ring of integers of the extension of $\mathbb{Q}(\psi, \phi)$ generated by the Fourier coefficients of f .

A general conjecture

Assume we have

- N, M - coprime, squarefree integers, $k > 2$ - integer.
- $E_k^{\psi, \phi}$ - new at level N ($\text{cond}(\psi) \cdot \text{cond}(\phi) = N$) with $\psi\phi = \chi$, $\tilde{\chi}$ is a lift of χ to modulus NM .
- $l > k + 1$, $l \nmid NM$ - prime of $\mathbb{Z}[\psi, \phi]$.
- λ - prime above l in the ring of integers of the extension of $\mathbb{Q}(\psi, \phi)$ generated by the Fourier coefficients of f .

Conjecture (Fretwell, R.)

There exists a newform $f \in S_k(\Gamma_1(NM), \tilde{\chi})$ such that

$$a_q(f) \equiv a_q(E_k^{\psi, \phi}) \pmod{\lambda}$$

for all primes $q \nmid NMl$, if and only if both of the following hold:

- ① $\text{ord}_l(L(1 - k, \psi^{-1}\phi) \prod_{p \in \mathcal{P}_M} (\psi(p) - \phi(p)p^k)) > 0$.
- ② $l \mid (\psi(p) - \phi(p)p^k)(\psi(p) - \phi(p)p^{k-2}) \quad \forall p \in \mathcal{P}_M$.

A general conjecture

Assume we have

- N, M - coprime, squarefree integers, $k > 2$ - integer.
- $E_k^{\psi, \phi}$ - new at level N ($\text{cond}(\psi) \cdot \text{cond}(\phi) = N$) with $\psi\phi = \chi$, $\tilde{\chi}$ is a lift of χ to modulus NM .
- $l > k + 1$, $l \nmid NM$ - prime of $\mathbb{Z}[\psi, \phi]$.
- λ - prime above l in the ring of integers of the extension of $\mathbb{Q}(\psi, \phi)$ generated by the Fourier coefficients of f .

Conjecture (Fretwell, R.)

There exists a newform $f \in S_k(\Gamma_1(NM), \tilde{\chi})$ such that

$$a_q(f) \equiv a_q(E_k^{\psi, \phi}) \pmod{\lambda}$$

for all primes $q \nmid NMl$, if and only if both of the following hold:

- ① $\text{ord}_l(L(1 - k, \psi^{-1}\phi) \prod_{p \in \mathcal{P}_M} (\psi(p) - \phi(p)p^k)) > 0$.
- ② $l \mid (\psi(p) - \phi(p)p^k)(\psi(p) - \phi(p)p^{k-2}) \quad \forall p \in \mathcal{P}_M$.

- ① Tells us that we will get a congruence modulo prime l with some eigenform f
- ② Gives us information about how “new” f is

A general conjecture (simplified)

How does the generalisation compare to Ramanujan's result?

A general conjecture (simplified)

How does the generalisation compare to Ramanujan's result?

- o We had a congruence between a cuspform Δ and Eisenstein series E_{12} .

A general conjecture (simplified)

How does the generalisation compare to Ramanujan's result?

- We had a congruence between a cuspform Δ and Eisenstein series E_{12} .
- We can find a congruence between a cuspform $f = \sum_{n \geq 1} a_n q^n$ (new at level NM) and an Eisenstein series $E_k^{\psi, \phi}$, (new at level N).

A general conjecture (simplified)

How does the generalisation compare to Ramanujan's result?

- We had a congruence between a cuspform Δ and Eisenstein series E_{12} .
- We can find a congruence between a cuspform $f = \sum_{n \geq 1} a_n q^n$ (new at level NM) and an Eisenstein series $E_k^{\psi, \phi}$, (new at level N).
- The congruence held modulo 691, this worked because 691 divided B_{12} (the constant coefficient of E_{12}).

A general conjecture (simplified)

How does the generalisation compare to Ramanujan's result?

- We had a congruence between a cuspform Δ and Eisenstein series E_{12} .
- We can find a congruence between a cuspform $f = \sum_{n \geq 1} a_n q^n$ (new at level NM) and an Eisenstein series $E_k^{\psi, \phi}$, (new at level N).
- The congruence held modulo 691, this worked because 691 divided B_{12} (the constant coefficient of E_{12}).
- The congruence holds modulo prime l , where l divides either the constant term of the Eisenstein series or an Euler product (with Euler factors at primes dividing M).
- To ensure f is new, l also has to divide another quantity at each prime dividing M .

Direct implication

Assume $a_q(f) \equiv \psi(q) + \phi(q)q^{k-1} \pmod{\lambda}$ for some newform $f \in S_k(\Gamma_1(NM), \tilde{\chi})$, λ a prime above l .

There is an l -adic Galois representation attached to f :

$$\rho_{f,l} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\overline{\mathbb{Z}_l})$$

with mod l reduction given by:

$$\overline{\rho}_{f,l} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\overline{\mathbb{F}_l}).$$

By the congruence condition, we then have:

$$\text{Tr}(\overline{\rho}_{f,l}) = \overline{\psi} + \overline{\phi}\chi_l^{k-1},$$

i.e. $\overline{\rho}_{f,l}$ has composition factors $\{\overline{\psi}, \overline{\phi}\chi_l^{k-1}\}$, where χ_l is the mod l cyclotomic character.

Direct implication

Assume $a_q(f) \equiv \psi(q) + \phi(q)q^{k-1} \pmod{\lambda}$ for some newform $f \in S_k(\Gamma_1(NM), \tilde{\chi})$, λ a prime above l .

There is an l -adic Galois representation attached to f :

$$\rho_{f,l} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\overline{\mathbb{Z}_l})$$

with mod l reduction given by:

$$\overline{\rho}_{f,l} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\overline{\mathbb{F}_l}).$$

By the congruence condition, we then have:

$$\text{Tr}(\overline{\rho}_{f,l}) = \overline{\psi} + \overline{\phi}\chi_l^{k-1},$$

i.e. $\overline{\rho}_{f,l}$ has composition factors $\{\overline{\psi}, \overline{\phi}\chi_l^{k-1}\}$, where χ_l is the mod l cyclotomic character.

Direct implication

Assume $a_q(f) \equiv \psi(q) + \phi(q)q^{k-1} \pmod{\lambda}$ for some newform $f \in S_k(\Gamma_1(NM), \tilde{\chi})$, λ a prime above l .

There is an l -adic Galois representation attached to f :

$$\rho_{f,l} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\overline{\mathbb{Z}_l})$$

with mod l reduction given by:

$$\overline{\rho}_{f,l} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\overline{\mathbb{F}_l}).$$

By the congruence condition, we then have:

$$\text{Tr}(\overline{\rho}_{f,l}) = \overline{\psi} + \overline{\phi}\chi_l^{k-1},$$

i.e. $\overline{\rho}_{f,l}$ has composition factors $\{\overline{\psi}, \overline{\phi}\chi_l^{k-1}\}$, where χ_l is the mod l cyclotomic character.

Direct implication

Assume $a_q(f) \equiv \psi(q) + \phi(q)q^{k-1} \pmod{\lambda}$ for some newform $f \in S_k(\Gamma_1(NM), \tilde{\chi})$, λ a prime above l .

There is an l -adic Galois representation attached to f :

$$\rho_{f,l} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\overline{\mathbb{Z}_l})$$

with mod l reduction given by:

$$\overline{\rho}_{f,l} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\overline{\mathbb{F}_l}).$$

By the congruence condition, we then have:

$$\text{Tr}(\overline{\rho}_{f,l}) = \overline{\psi} + \overline{\phi}\chi_l^{k-1},$$

i.e. $\overline{\rho}_{f,l}$ has composition factors $\{\overline{\psi}, \overline{\phi}\chi_l^{k-1}\}$, where χ_l is the mod l cyclotomic character.

Let p be a prime dividing M . Then the composition factors of $\bar{\rho}_{f,l}$ “locally at p ” are:

$$\{\mu\chi_l^{k/2}, \mu\chi_l^{k/2-1}\}$$

where μ is the unramified character mapping the Frobenius at p to $a_p(f)/p^{k/2-1} \pmod{\lambda}$.

Equating the two sets of composition factors locally at p :

$$\{\bar{\psi}, \bar{\phi}\chi_l^{k-1}\} = \{\mu\chi_l^{k/2}, \mu\chi_l^{k/2-1}\},$$

which leaves us with two cases:

(A) $\bar{\psi} = \mu\chi_l^{k/2}, \bar{\phi}\chi_l^{k-1} = \mu\chi_l^{k/2-1}$.

(B) $\bar{\psi} = \mu\chi_l^{k/2-1}, \bar{\phi}\chi_l^{k-1} = \mu\chi_l^{k/2}$.

Let p be a prime dividing M . Then the composition factors of $\bar{\rho}_{f,l}$ “locally at p ” are:

$$\{\mu\chi_l^{k/2}, \mu\chi_l^{k/2-1}\}$$

where μ is the unramified character mapping the Frobenius at p to $a_p(f)/p^{k/2-1} \pmod{\lambda}$.

Equating the two sets of composition factors locally at p :

$$\{\bar{\psi}, \bar{\phi}\chi_l^{k-1}\} = \{\mu\chi_l^{k/2}, \mu\chi_l^{k/2-1}\},$$

which leaves us with two cases:

(A) $\bar{\psi} = \mu\chi_l^{k/2}, \bar{\phi}\chi_l^{k-1} = \mu\chi_l^{k/2-1}$.

(B) $\bar{\psi} = \mu\chi_l^{k/2-1}, \bar{\phi}\chi_l^{k-1} = \mu\chi_l^{k/2}$.

Let p be a prime dividing M . Then the composition factors of $\bar{\rho}_{f,l}$ “locally at p ” are:

$$\{\mu\chi_l^{k/2}, \mu\chi_l^{k/2-1}\}$$

where μ is the unramified character mapping the Frobenius at p to $a_p(f)/p^{k/2-1} \pmod{\lambda}$.

Equating the two sets of composition factors locally at p :

$$\{\bar{\psi}, \bar{\phi}\chi_l^{k-1}\} = \{\mu\chi_l^{k/2}, \mu\chi_l^{k/2-1}\},$$

which leaves us with two cases:

(A) $\bar{\psi} = \mu\chi_l^{k/2}, \bar{\phi}\chi_l^{k-1} = \mu\chi_l^{k/2-1}$.

(B) $\bar{\psi} = \mu\chi_l^{k/2-1}, \bar{\phi}\chi_l^{k-1} = \mu\chi_l^{k/2}$.

Let p be a prime dividing M . Then the composition factors of $\bar{\rho}_{f,l}$ “locally at p ” are:

$$\{\mu\chi_l^{k/2}, \mu\chi_l^{k/2-1}\}$$

where μ is the unramified character mapping the Frobenius at p to $a_p(f)/p^{k/2-1} \pmod{\lambda}$.

Equating the two sets of composition factors locally at p :

$$\{\bar{\psi}, \bar{\phi}\chi_l^{k-1}\} = \{\mu\chi_l^{k/2}, \mu\chi_l^{k/2-1}\},$$

which leaves us with two cases:

- (A) $\bar{\psi} = \mu\chi_l^{k/2}$, $\bar{\phi}\chi_l^{k-1} = \mu\chi_l^{k/2-1}$.
- (B) $\bar{\psi} = \mu\chi_l^{k/2-1}$, $\bar{\phi}\chi_l^{k-1} = \mu\chi_l^{k/2}$.

Direct implication

Evaluating at $p \mid M$ in both cases:

$$(A) \quad \left. \begin{array}{l} \psi(p) \equiv \mu(p)p^{k/2} \pmod{l} \\ \phi(p)p^{k-1} \equiv \mu(p)p^{k/2-1} \pmod{l} \end{array} \right\} \implies \psi(p) - \phi(p)p^k \equiv 0 \pmod{l}.$$

$$(B) \quad \left. \begin{array}{l} \psi(p) \equiv \mu(p)p^{k/2-1} \pmod{l} \\ \phi(p)p^{k-1} \equiv \mu(p)p^{k/2} \pmod{l} \end{array} \right\} \implies \psi(p) - \phi(p)p^{k-2} \equiv 0 \pmod{l}.$$

Also, since $\mu(p) \equiv a_p(f)/p^{k/2-1} \pmod{\lambda}$, we have $a_p(f) \equiv \psi(p) \pmod{\lambda}$ in this case.

To summarise, we have that for all p dividing M , we require one of the following:

- (A) $\psi(p) - \phi(p)p^k \equiv 0 \pmod{l}$.
- (B) $\psi(p) - \phi(p)p^{k-2} \equiv 0 \pmod{l}$ and $a_p(f) \equiv \psi(p) \pmod{\lambda}$.

Condition 2

$$l \mid (\psi(p) - \phi(p)p^k)(\psi(p) - \phi(p)p^{k-2}) \quad \forall p \in \mathcal{P}_M.$$

This proves the necessity of Condition (2) in the statement of the theorem.

Direct implication

Evaluating at $p \mid M$ in both cases:

$$(A) \quad \left. \begin{array}{l} \psi(p) \equiv \mu(p)p^{k/2} \pmod{l} \\ \phi(p)p^{k-1} \equiv \mu(p)p^{k/2-1} \pmod{l} \end{array} \right\} \implies \psi(p) - \phi(p)p^k \equiv 0 \pmod{l}.$$

$$(B) \quad \left. \begin{array}{l} \psi(p) \equiv \mu(p)p^{k/2-1} \pmod{l} \\ \phi(p)p^{k-1} \equiv \mu(p)p^{k/2} \pmod{l} \end{array} \right\} \implies \psi(p) - \phi(p)p^{k-2} \equiv 0 \pmod{l}.$$

Also, since $\mu(p) \equiv a_p(f)/p^{k/2-1} \pmod{\lambda}$, we have $a_p(f) \equiv \psi(p) \pmod{\lambda}$ in this case.

To summarise, we have that for all p dividing M , we require one of the following:

- (A) $\psi(p) - \phi(p)p^k \equiv 0 \pmod{l}$.
- (B) $\psi(p) - \phi(p)p^{k-2} \equiv 0 \pmod{l}$ and $a_p(f) \equiv \psi(p) \pmod{\lambda}$.

Condition 2

$$l \mid (\psi(p) - \phi(p)p^k)(\psi(p) - \phi(p)p^{k-2}) \quad \forall p \in \mathcal{P}_M.$$

This proves the necessity of Condition (2) in the statement of the theorem.

Direct implication

Evaluating at $p \mid M$ in both cases:

$$(A) \left. \begin{array}{l} \psi(p) \equiv \mu(p)p^{k/2} \pmod{l} \\ \phi(p)p^{k-1} \equiv \mu(p)p^{k/2-1} \pmod{l} \end{array} \right\} \implies \psi(p) - \phi(p)p^k \equiv 0 \pmod{l}.$$

$$(B) \left. \begin{array}{l} \psi(p) \equiv \mu(p)p^{k/2-1} \pmod{l} \\ \phi(p)p^{k-1} \equiv \mu(p)p^{k/2} \pmod{l} \end{array} \right\} \implies \psi(p) - \phi(p)p^{k-2} \equiv 0 \pmod{l}.$$

Also, since $\mu(p) \equiv a_p(f)/p^{k/2-1} \pmod{\lambda}$, we have $a_p(f) \equiv \psi(p) \pmod{\lambda}$ in this case.

To summarise, we have that for all p dividing M , we require one of the following:

- (A) $\psi(p) - \phi(p)p^k \equiv 0 \pmod{l}$.
- (B) $\psi(p) - \phi(p)p^{k-2} \equiv 0 \pmod{l}$ and $a_p(f) \equiv \psi(p) \pmod{\lambda}$.

Condition 2

$$l \mid (\psi(p) - \phi(p)p^k)(\psi(p) - \phi(p)p^{k-2}) \quad \forall p \in \mathcal{P}_M.$$

This proves the necessity of Condition (2) in the statement of the theorem.

Direct implication

We can do something similar for primes p dividing N , which in this case gives us an equivalence of characters:

$$\{\mu_1 \chi_l^{(k-1)/2}, \mu_2 \chi_l^{(k-1)/2}\} = \{\bar{\psi}, \bar{\phi} \chi_l^{k-1}\}.$$

We also have $a_p(f) \equiv p^{(k-1)/2}(\mu_1(p) + \mu_2(p)) \pmod{\lambda}$, hence $a_p(f) \equiv \psi(p) + \phi(p)p^{k-1} \pmod{\lambda}$.

Note that if any prime $p \mid M$ satisfies case (A), then Condition (1) follows immediately and we are done. So, assume that for all $p \mid M$, we are in case (B):

$$\psi(p) - \phi(p)p^{k-2} \equiv 0 \pmod{l},$$

$$a_p(f) \equiv \psi(p) \pmod{\lambda}.$$

Need to show:

Condition 1

$$\text{ord}_l(L(1-k, \psi^{-1}\phi) \prod_{p \in \mathcal{P}_M} (\psi(p) - \phi(p)p^k)) > 0.$$

Direct implication

We can do something similar for primes p dividing N , which in this case gives us an equivalence of characters:

$$\{\mu_1 \chi_l^{(k-1)/2}, \mu_2 \chi_l^{(k-1)/2}\} = \{\bar{\psi}, \bar{\phi} \chi_l^{k-1}\}.$$

We also have $a_p(f) \equiv p^{(k-1)/2}(\mu_1(p) + \mu_2(p)) \pmod{\lambda}$, hence $a_p(f) \equiv \psi(p) + \phi(p)p^{k-1} \pmod{\lambda}$.

Note that if any prime $p \mid M$ satisfies case (A), then Condition (1) follows immediately and we are done. So, assume that for all $p \mid M$, we are in case (B):

$$\psi(p) - \phi(p)p^{k-2} \equiv 0 \pmod{l},$$

$$a_p(f) \equiv \psi(p) \pmod{\lambda}.$$

Need to show:

Condition 1

$$\text{ord}_l(L(1-k, \psi^{-1}\phi) \prod_{p \in \mathcal{P}_M} (\psi(p) - \phi(p)p^k)) > 0.$$

Direct implication

Let

$$E = \left[\prod_{p \in \mathcal{P}_M} (T_p - \phi(p)p^{k-1}) \right] \alpha_M E_k^{\psi, \phi},$$

where $\alpha_M f(z) := f(Mz)$ and T_p is the p^{th} Hecke operator.

Fact 1

E is a normalised eigenform with Fourier coefficient at prime p given by:

$$a_p(E) = \begin{cases} \psi(p) & \text{if } p \mid M \\ \psi(p) + \phi(p)p^{k-1} & \text{otherwise} \end{cases}$$

With a bit more work, this gives

$$a_n(E) \equiv a_n(f) \pmod{\lambda} \quad \forall n$$

In particular, since f is a cusp form, this tells us that E must vanish at all cusps.

Direct implication

Let

$$E = \left[\prod_{p \in \mathcal{P}_M} (T_p - \phi(p)p^{k-1}) \right] \alpha_M E_k^{\psi, \phi},$$

where $\alpha_M f(z) := f(Mz)$ and T_p is the p^{th} Hecke operator.

Fact 1

E is a normalised eigenform with Fourier coefficient at prime p given by:

$$a_p(E) = \begin{cases} \psi(p) & \text{if } p \mid M \\ \psi(p) + \phi(p)p^{k-1} & \text{otherwise} \end{cases}$$

With a bit more work, this gives

$$a_n(E) \equiv a_n(f) \pmod{\lambda} \quad \forall n$$

In particular, since f is a cusp form, this tells us that E must vanish at all cusps.

Direct implication

Let

$$E = \left[\prod_{p \in \mathcal{P}_M} (T_p - \phi(p)p^{k-1}) \right] \alpha_M E_k^{\psi, \phi},$$

where $\alpha_M f(z) := f(Mz)$ and T_p is the p^{th} Hecke operator.

Fact 1

E is a normalised eigenform with Fourier coefficient at prime p given by:

$$a_p(E) = \begin{cases} \psi(p) & \text{if } p \mid M \\ \psi(p) + \phi(p)p^{k-1} & \text{otherwise} \end{cases}$$

With a bit more work, this gives

$$a_n(E) \equiv a_n(f) \pmod{\lambda} \quad \forall n$$

In particular, since f is a cusp form, this tells us that E must vanish at all cusps.

Direct implication

Fact 2

We can choose a $\gamma = \begin{pmatrix} a & \beta \\ b & \delta \end{pmatrix} \in SL_2(\mathbb{Z})$ such that $E[\gamma]_k$ has constant term:

$$-\frac{g(\psi\phi^{-1})}{g(\phi)} \frac{\phi^{-1}(a)\psi\left(\frac{-b}{v}\right)\psi^{-1}(M)}{u^k} L(1-k, \psi^{-1}\phi) \left(\prod_{p \in \mathcal{P}_M} (\psi(p) - \phi(p)p^{k-1}) \right)$$

However, $\frac{g(\psi\phi^{-1})}{g(\phi)}$, $\phi^{-1}(a)$, $\psi\left(\frac{-b}{v}\right)$ and $\psi^{-1}(M)$ are units in $\mathbb{Z}[\psi, \phi]$.

Furthermore, $l \nmid u$ since $u \mid N$ and by assumption $l \nmid N$. Hence, l must divide

$$L(1-k, \psi^{-1}\phi) \left(\prod_{p \in \mathcal{P}_M} (\psi(p) - \phi(p)p^{k-1}) \right).$$

Condition (1) follows easily from this, so we are done.

Direct implication

Fact 2

We can choose a $\gamma = \begin{pmatrix} a & \beta \\ b & \delta \end{pmatrix} \in SL_2(\mathbb{Z})$ such that $E[\gamma]_k$ has constant term:

$$-\frac{g(\psi\phi^{-1})}{g(\phi)} \frac{\phi^{-1}(a)\psi\left(\frac{-b}{v}\right)\psi^{-1}(M)}{u^k} L(1-k, \psi^{-1}\phi) \left(\prod_{p \in \mathcal{P}_M} (\psi(p) - \phi(p)p^{k-1}) \right)$$

However, $\frac{g(\psi\phi^{-1})}{g(\phi)}$, $\phi^{-1}(a)$, $\psi\left(\frac{-b}{v}\right)$ and $\psi^{-1}(M)$ are units in $\mathbb{Z}[\psi, \phi]$.

Furthermore, $l \nmid u$ since $u \mid N$ and by assumption $l \nmid N$. Hence, l must divide

$$L(1-k, \psi^{-1}\phi) \left(\prod_{p \in \mathcal{P}_M} (\psi(p) - \phi(p)p^{k-1}) \right).$$

Condition (1) follows easily from this, so we are done.

Reverse implication

- For the reverse implication, we can prove the case where $M = p$ prime.
- A result of Spencer (2018) gives us an eigenform f which satisfies the congruence.
- If this is a newform then we're done, so assume f arises from a modular form $f_0 \in S_k^{new}(\Gamma_0(N), \chi)$.

Reverse implication

- For the reverse implication, we can prove the case where $M = p$ prime.
- A result of Spencer (2018) gives us an eigenform f which satisfies the congruence.
- If this is a newform then we're done, so assume f arises from a modular form $f_0 \in S_k^{new}(\Gamma_0(N), \chi)$.

To obtain a level Np newform, we use the following:

Theorem (Diamond 1991)

For $p \nmid N$, the following are equivalent:

- 1 We have $f_0 \in S_k(\Gamma_0(N), \chi)$ satisfying

$$a_p(f_0)^2 \equiv \chi(p)p^{k-2}(1 + p^2) \pmod{\lambda}.$$

- 2 There is a p -newform $f_1 \in S_k(\Gamma_0(Np), \chi)$ satisfying

$$a_q(f_1) \equiv a_q(f_0) \equiv \psi(q) + \phi(q)q^{k-1} \pmod{\lambda'} \quad \forall \text{ primes } q \nmid Np\ell$$

Consequences/ future work

- Our results provide evidence for a case of the Bloch-Kato conjecture - a conjecture in algebraic number theory relating L-values and arithmetic of Galois modules.

Consequences/ future work

- Our results provide evidence for a case of the Bloch-Kato conjecture - a conjecture in algebraic number theory relating L-values and arithmetic of Galois modules.
- In particular, our results are expected to produce 'new' elements in a Bloch-Kato Selmer group.

Consequences/ future work

- Our results provide evidence for a case of the Bloch-Kato conjecture - a conjecture in algebraic number theory relating L-values and arithmetic of Galois modules.
- In particular, our results are expected to produce 'new' elements in a Bloch-Kato Selmer group.
- We are also working on generalising our results to Hilbert modular forms.

Thank you!