Newform Eisenstein congruences of local origin

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A motivating example

Take the discriminant function

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So, we have

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$\tau(n)$	1	-24	252	-1472	4830	-6048

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Ramanujan's discovery

$\tau(n) \equiv \sigma_{11}(n) \mod 691$

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Proof (Sketch)

Consider the normalised, weight 12 Eisenstein series,

$$E_{12}(z) = \frac{B_{12}}{24} + \sum_{n \geq 1} \sigma_{11}(n) q^n \in M_{12}(SL_2(\mathbb{Z}))$$

and note that the discriminant function is a weight 12 cusp form,

$$\Delta(z) = \sum_{n \geq 1} \tau(n) q^n \in S_{12}(SL_2(\mathbb{Z})) \subset M_{12}(SL_2(\mathbb{Z})).$$

Proof (Sketch)

Where does 691 appear?

- 691 divides B_{12}, so the constant coefficient of E_{12} vanishes modulo 691. E_{12} and Δ are both modular forms of weight 12 whose constant terms vanish modulo 691 - "cusp forms mod 691".
- The dimension formula tells us there is only one cusp form of weight 12, so we conclude that E_{12} and Δ must be the same cusp form mod 691.

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There exists $[a] \in \mathrm{Cl}(\mathbb{Q}(\mu_{691}))[691]$ satisfying:

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where $\chi_{691} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{F}_{691}^*$ is the mod 691 cyclotomic character.

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More generally, Eisenstein congruences were used to prove the Herbrand-Ribet theorem:
For $2 \leq k \leq p-3$ even:

$$\mathrm{ord}_p(B_k) > 0 \iff \exists \text{ element in the } \chi_p^{1-k} \text{ eigenspace of } \mathrm{Cl}(\mathbb{Q}(\mu_p))[p].$$

In particular, Ribet proved the following:

Ribet's converse to Herbrand's theorem

For $2 \leq k \leq p-3$ even:
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We're going to break down his proof into steps, but first let's define a modular form...

Modular forms of level N

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

Let $M_k(\Gamma_1(N), \chi)$ be the space of modular forms of weight $k \geq 2$, level N and Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ with conductor N.

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$f \in M_k(\Gamma_1(N), \chi)$ if:

- f is holomorphic on the complex upper half plane
- f satisfies:

$$f[\gamma]_k := (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = \chi(d) f(z)$$

  for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.
- $f[\alpha]_k$ must be holomorphic at ∞ for all $\alpha \in SL_2(\mathbb{Z})$, i.e. at all cusps.

Generalised Eisenstein series

Take Dirichlet characters ψ, ϕ satisfying $\psi\phi = \chi$. Then we have that the Eisenstein series, $E_k^{\psi,\phi} \in M_k(\Gamma_0(N), \chi)$, where:

$$E_k^{\psi,\phi}(z) = \frac{1}{2}\delta(\psi)L(1-k, \psi^{-1}\phi) + \sum_{n=1}^{\infty} \sigma_{k-1}^{\psi,\phi}(n)q^n,$$

$\delta(\psi) = 1$ if ψ is the trivial mod 1 character, 0 otherwise and

$$\sigma_{k-1}^{\psi,\phi}(n) = \sum_{d|n, d>0} \psi(n/d)\phi(d)d^{k-1}$$

is the generalised power divisor series.

Newforms

- If we were to take two modular forms, f and g of level N/d, we could raise them to a modular form of level N with the map:

$$f + g[\alpha_d]_k \quad \text{where } \alpha_d = \left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right).$$

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- The new subspace is the orthogonal complement to this space w.r.t. the Petersson innner product. A modular form is **new** at level N if it lies in the new subspace.
- An **eigenform** is an eigenvector for all the Hecke operators, T_p, with $p \nmid N$.
- $f = \sum_{n \geq 0} a_n(f) q^n \in M_k(\Gamma_0(N), \chi)$ is **normalised** if $a_1(f) = 1$.
- We say $f \in M_k(\Gamma_0(N), \chi)$ is a **newform** if f is a normalised eigenform which is new at level N.

Steps of the proof

Ribet's converse to Herbrand's theorem

For $2 \leq k \leq p-3$ even:
If $\text{ord}_p(B_k) > 0$ then there exists an element $[a] \in \text{Cl}(\mathbb{Q}(\mu_p))[p]$ satisfying

$$\sigma \cdot [a] = \chi_p(\sigma)^{1-k}[a] \text{ for all } \sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}).$$

1. **Bernoulli number**, B_k
2. **Eisenstein series**, $E_k = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n \in M_k(SL_2(\mathbb{Z}))$
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5. **Ideal class groups**

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$2 \rightarrow 3$: There is a congruence between Eisenstein series E_k and cuspform f modulo p

Show: $E_k \equiv f$ mod $\mathfrak{p}$, $\mathfrak{p} \mid p$ in field K_f generated by the coefficients of f

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Show: $E_k \equiv f \bmod \mathfrak{p}$, $\mathfrak{p} \mid p$ in field K_f generated by the coefficients of f

- Take $E_{2,\varepsilon} = \frac{L(-1,\varepsilon)}{2} + \sum_{n=1}^{\infty} \left(\sum_{0<d|n} \varepsilon(d)d \right) q^n \in M_2(\Gamma_1(p), \varepsilon)$.

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  Eisenstein series of weight 2, level p and character ε
- Fix a prime ideal $\mathfrak{p} \mid p$ in $\mathbb{Q}(\mu_{p-1})$ and let $\omega : (\mathbb{Z}/p\mathbb{Z})^* \xrightarrow{\sim} \mu_{p-1}$ be the unique Dirichlet character which satisfies

$$\omega(d) \equiv d \pmod{\mathfrak{p}} \ \forall\ d \in \mathbb{Z}.$$

- Then $E_{2,\omega^{k-2}}$ has a $\mathfrak{p}$-integral q-expansion in $\mathbb{Q}(\mu_{p-1})$ and

$$E_{2,\omega^{k-2}} \equiv E_k \pmod{\mathfrak{p}}$$

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- Ribet showed: $\exists\ g \in M_2(\Gamma_1(p), \omega^{k-2})$ whose q-expansion coefficients are $\mathfrak{p}$-integers in $\mathbb{Q}(\mu_{p-1})$ and whose constant term is 1.
- Take $f' = E_{2,\omega^{k-2}} - c \cdot g$ where c is the constant term of $E_{k,\omega^{k-2}}$
- Then if $\mathrm{ord}_p(B_k) > 0$, $f' \equiv E_k \equiv E_{2,\omega^{k-2}} \pmod{\mathfrak{p}}$

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  f is a mod $\mathfrak{p}$-eigenform since it is congruent to E_k mod $\mathfrak{p}$.
- Use Deligne-Serre to get $f \equiv f' \pmod{\tilde{\mathfrak{p}}}$ for some $\tilde{\mathfrak{p}} \mid \mathfrak{p}$ in $K_f(\mu_{p-1})$ for f an eigenform
- Can check f is cuspidal, so f is the required cuspform

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   ↓ There is a residually reducible mod p Galois rep attached to f
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$3 \to 4$: There is a residually reducible mod p Galois rep attached to f

- By a theorem of Deligne, for prime p, there is a continuous, irreducible Galois rep. attached to a cusp form $f \in S_2(\Gamma_1(p), \omega^{k-2})$

$$\rho_{f,p} : G_{\mathbb{Q}} \to GL_2(\mathbb{Q}_p)$$

which is unramified for $q \neq p$ and satisfies

$$\mathrm{Tr}(\rho_{f,p}(\mathrm{Frob}_q)) = a_q(f), \qquad \det(\rho_{f,p}(\mathrm{Frob}_q)) = \omega^{k-2}(q)q$$

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- We can reduce this representation modulo p to get

$$\bar{\rho}_{f,p} : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_p)$$

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- Congruence condition on f tells us that the semisimplification of this must be
$$\bar{\rho}_{f,p}^{ss} \sim 1 \oplus \chi_p^{k-1}$$
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- Ribet shows we can always choose a basis for $\rho_{f,p}$ such that

$$\bar{\rho}_{f,p} \sim \begin{pmatrix} 1 & * \\ 0 & \chi_p^{k-1} \end{pmatrix}$$

i.e. evaluated at $\sigma \in G_{\mathbb{Q}}$,

$$\bar{\rho}_{f,p}(\sigma) = \begin{pmatrix} 1 & \bar{b}(\sigma) \\ 0 & \chi_p^{k-1}(\sigma) \end{pmatrix}$$

and $\rho_{f,p}$ is everywhere unramified.

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$4 \to 5$: There is a non-trivial element in $\mathrm{Cl}(\mathbb{Q}(\mu_p))[p]$

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- This is equivalent to the existence of a field extension $E/\mathbb{Q}$ with (writing $F = \mathbb{Q}(\mu_p)$):
    - $E \supset F$ and $\mathrm{Gal}(E/F) \cong \mathbb{F}_p$
    - E everywhere unramified
    - the action of $\mathrm{Gal}(F/\mathbb{Q})$ on $\mathrm{Gal}(E/F)$ via conjugation is given by

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    $$\sigma\tau\sigma^{-1} = \chi_p^{1-k}(\sigma)\tau$$

    for $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$, $\tau \in \mathrm{Gal}(E/F)$
- Class Field Theory then gives us existence of a non-trivial element in the χ_p^{k-1}-eigenspace of $\mathrm{Cl}(\mathbb{Q}(\mu_p))[p]$

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Goal: Generalise to newform congruences with non-trivial character and lift by prime level.

Assume we have

- N - squarefree, p prime, $(N, p) = 1$, $k > 2$ - integer.
- $E_k^{\psi,\phi}$ - new at level N (cond$(\psi)\cdot$cond$(\phi) = N$) with $\psi\phi = \chi$, $\tilde{\chi}$ is a lift of χ to modulus Np.
- $l > k+1$, $l \nmid \varphi(N)Np$, l prime.
- λ - prime above l in the ring of integers of the extension of $\mathbb{Q}(\psi, \phi)$ generated by the Fourier coefficients of f.

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- λ - prime above l in the ring of integers of the extension of $\mathbb{Q}(\psi,\phi)$ generated by the Fourier coefficients of f.

Theorem (Fretwell, R., 2024)

There exists a newform $f\in S_k(\Gamma_1(Np),\tilde{\chi})$ such that

$$a_q(f)\equiv a_q(E_k^{\psi,\phi}) \pmod{\lambda}$$

for all primes $q\nmid Npl$, if and only if both of the following hold for some $\lambda'\mid l$ in $\mathbb{Z}[\psi,\phi]$:

1. $\mathrm{ord}_{\lambda'}(L(1-k,\psi^{-1}\phi)(\psi(p)-\phi(p)p^k))>0$.
2. $\mathrm{ord}_{\lambda'}((\psi(p)-\phi(p)p^k)(\psi(p)-\phi(p)p^{k-2}))>0$.

Assume we have

- N - squarefree, p prime, $(N, p) = 1$, $k > 2$ - integer.
- $E_k^{\psi,\phi}$ - new at level N (cond$(\psi)\cdot$cond$(\phi) = N$) with $\psi\phi = \chi$, $\tilde{\chi}$ is a lift of χ to modulus Np.
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1. Tells us that we will get a congruence modulo prime l with some eigenform f
2. Gives us information about how "new" f is

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- To ensure f is new, l also has to divide another quantity depending on p.

(1)+(2) $\implies$ newform congruence

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Assume we have conditions (1) and (2) from the theorem:

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- Construct modular form $E^{(\psi)}(z) = E_k^{\psi,\phi}(z) - \psi(p)E_k^{\psi,\phi}(pz)$. This has constant term at the cusps given by either:

$$a_0(E^{(\psi)}[\gamma]_k) = c_\gamma L(1-k,\psi^{-1}\phi)(\psi(p)-\phi(p)p^k)$$

  or $a_0(E^{(\psi)}[\gamma]_k) = 0$.
  - Since it is also an eigenform, this is a "mod λ' cusp form"
- Deligne-Serre gives existence of eigenform f such that

$$f \equiv E^{(\psi)} \equiv E_k^{\psi,\phi} \pmod{\lambda}$$

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- To show f is a newform, first note we have minimum possible level $f \in S_k(\Gamma_1(N),\chi)$ since χ has conductor N.
- Assuming f is old at level Np, condition (2) is exactly what we need to use Diamond's level lifting lemma, which ensures there exists a newform $f_1 \in S_k(\Gamma_1(Np),\chi)$ satsfying the congruence condition.

Consequences

- The congruence gives us a λ-adic Galois rep $\rho_{f,\lambda}$ which when taken mod λ has semisimplification

$$\bar{\rho}_{f,\lambda}^{ss} \sim \bar{\psi} \oplus \bar{\phi}\chi_l^{k-1}$$

- We can choose the basis for $\rho_{f,\lambda}$ in such a way that $\bar{\rho}_{f,\lambda}$ is realised on an $\mathbb{F}_\lambda$-vector space V such that

$$0 \to \mathbb{F}_\lambda(1-k)(\phi) \xrightarrow{\iota} V \xrightarrow{\pi} \mathbb{F}_\lambda(\psi) \to 0$$

  is a non-split extension of $\mathbb{F}_\lambda[G_{\mathbb{Q}}]$-modules.
- Twist by ψ^{-1} gives a non-split extension $V(\psi^{-1})$ which defines a non-trivial class $c \in H^1(\mathbb{Q}, \mathbb{F}_\lambda(1-k)(\psi^{-1}\phi))$.

Consequences

- Taking $C_{k,l}^{\psi,\phi} = (\mathbb{Q}_l/\mathbb{Z}_l)(1-k)(\psi^{-1}\phi)$, the congruence implies $\exists\ c' \in H^1_{\mathcal{P}_{Np}}(\mathbb{Q}, C_{k,l}^{\psi,\phi})$.

  Classes of $H^1(\mathbb{Q}, C_{k,l}^{\psi,\phi})$ for which local restrictions for primes $q \nmid Np$ lie in Bloch-Kato Selmer group $H^1_f(\mathbb{Q}_q, C_{k,l}^{\psi,\phi})$
- Using this, we can recover evidence of a case of the Bloch-Kato conjecture which was proven by Huber and Kings.

Thank you!