Why I care about modular forms

Underrepresented genders in mathematics, 8th November 2023

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RAMANUJAN'S CONGRUENCE

- Consider the product:

$$q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n$$

- Now take the following function:

$$\sigma_{11}(n) = \sum_{d|n} d^{11}$$

- Let's compare the first few values…

How about mod 691?

$\boldsymbol{n}$	1	2	3	4	5	6
$\boldsymbol{\sigma_{11}(n)}$ mod 691	1	667	252	601	684	171
$\boldsymbol{\tau(n)}$ mod 691	1	667	252	601	684	171

RAMANUJAN'S CONGRUENCE

Ramanujan's observation

$$\sigma_{11}(n) \equiv \tau(n) \bmod 691$$

This seems surprising, but it can be explained using the theory of modular forms..

MODULAR FORM DEFINITION

Definition

We say $f: \mathcal{H} \rightarrow \mathbb{C}$ is a modular form if:

- f is holomorphic on the upper half plane $\mathcal{H}$
- f is holomorphic at ∞
- $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

Let's dissect what these conditions mean…

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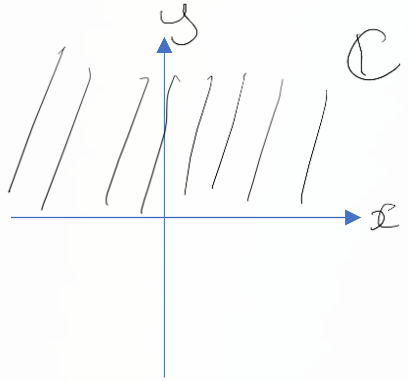
$\mathcal{H} = \{x + iy \in \mathbb{C} : y > 0\}$



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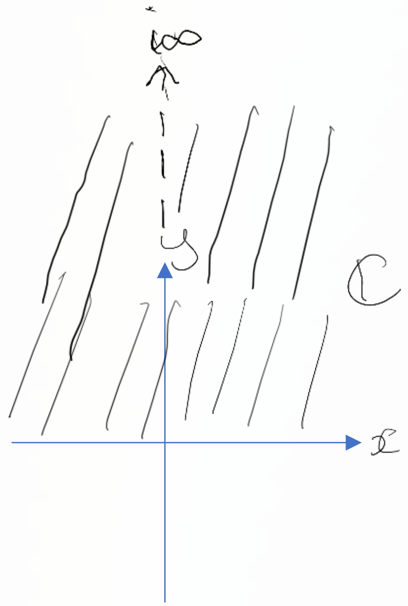

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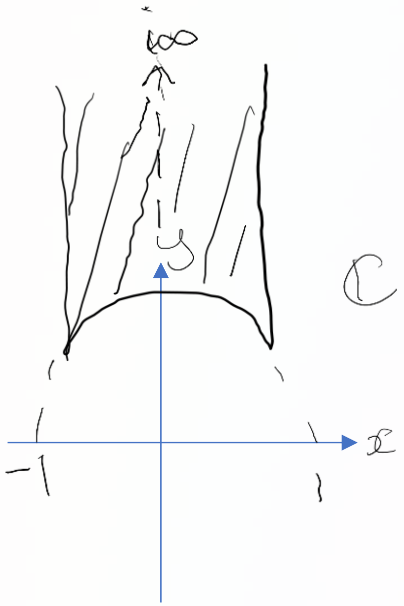
$SL_2(\mathbb{Z}) = \text{Span}\left\{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\}$, this gives:

- $f(z+1) = f(z)$
- $f\left(\frac{-1}{z}\right) = z^k f(z)$, we call $k \geq 2$ the **weight** (k even)



CONSEQUENCES OF MODULAR FORM DEFINITION

- Since $f(z+1) = f(z)$ we can write f as a Fourier series expansion:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n q^n$$

where $q = e^{2\pi i z}$.

  - This works since $e^{2\pi i z} = e^{2\pi i (z+1)}$.
  - We call the a_n Fourier coefficients.

- Also, since f is holomorphic, we have no negative coefficients:

$$f(z) = \sum_{n=0}^{\infty} a_n q^n$$

- If f and g are modular forms of weights k and l, then using the transformation formula, fg is a modular form of weight $k + l$.

EXAMPLES OF MODULAR FORMS

$\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n$ is a weight 12 modular form.

$E_k(z) = 1 + \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$, $\sigma_k\,(n) = \sum_{d|n} d^{k-1}$ is a weight k modular form.

In particular, consider…

$$E_4(z) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n = 1 + 240q + \cdots$$

Weight 4

$$E_6(z) = 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n = 1 - 504q + \cdots$$

Weight 6

$$\tilde{E}_{12}(z) = \frac{691}{65520} + \sum_{n=1}^{\infty}\sigma_{11}(n)q^n = \frac{691}{65520} + q + \cdots$$

Weight 12

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We wanted to show $\sigma_{11}(n) \equiv \tau(n) \bmod 691$, so we now need to show that $\tilde{E}_{12} \equiv \Delta \pmod{691}$.

Fact

The space of modular forms of weight k is spanned by $E_4^i E_6^j$, for $4i + 6j = k$.

- In other words, we can write any modular form of weight k as a linear combination of products of E_4 and E_6.

Let's look at weight 12:
There are only two i, j pairs such that $4i + 6j = 12$:
$i = 3, j = 0$ and $i = 0, j = 2$.
So modular forms of weight 12 can all be written in the form:

$$aE_4^3 + bE_6^2 = (a + b) + (3 * 240a - 2 * 504b)q + \cdots$$

Or taking two forms of weight 12, say Δ and $\tilde{E}_{12}$, we can write:

$$\Delta = \alpha E_6^2 + \beta \tilde{E}_{12} = \left(\alpha + \frac{691}{65520}\beta\right) + (-1008\alpha + \beta)q + \cdots$$

Comparing constant and 1st coefficients, we get:

$$\begin{cases} \alpha + \frac{691}{65520}\beta = 0 \\ -1008\alpha + \beta = 1 \end{cases} \Rightarrow \alpha = -\frac{691}{762048}, \beta = \frac{65}{756}$$

- From the previous slide, we have

$$\Delta = -\frac{691}{762048} E_6^2 + \frac{65}{756}\ \tilde{E}_{12}$$

- Clearing denominators, we get:

$$762048\Delta = -691\, E_6^2 + 65520\, \tilde{E}_{12}$$

$$(1008 * 691 + 65520)\Delta = -691\, E_6^2 + 65520\, \tilde{E}_{12}$$

- Now, working mod 691:

$$65520\Delta \equiv 65520\, \tilde{E}_{12} \pmod{691}$$

$$\Longrightarrow \Delta \equiv \tilde{E}_{12} \pmod{691}$$

$$\Longrightarrow \tau(n) = \sigma_{11}(n) \pmod{691} \text{ for all } n.$$

We’ve solved it!